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Giulio Guerrieri[1,2], Luc Pellissier[3], and Lorenzo Tortora de Falco[1]

1 **Dipartimento di Matematica e Fisica, Università Roma Tre, Rome, Italy**
`{gguerrieri,tortora}@uniroma3.it`
2 **I2M, UMR 7373, Aix-Marseille Université, F-13453 Marseille, France**
3 **LIPN, UMR 7030, Université Paris 13, Sorbonne Paris Cité**
F-93430 Villetaneuse, France
`luc.pellissier@lipn.univ-paris13.fr`

Abstract

We show that every connected Multiplicative Exponential Linear Logic (MELL) proof-structure (with or without cuts) is uniquely determined by a well-chosen element of its Taylor expansion: the one obtained by taking two copies of the content of each box. As a consequence, the relational model is injective with respect to connected MELL proof-structures.

1 Introduction

Given a syntax $\mathcal{S}$ endowed with some rewrite rules, and given a denotational model $\mathcal{D}$ for $\mathcal{S}$ (*i.e.* a semantics which gives to any term t of $\mathcal{S}$ an interpretation $[\![t]\!]_{\mathcal{D}}$ that is invariant under the rewrite rules), we say that $\mathcal{D}$ is *injective* with respect to $\mathcal{S}$ if, for any two normal terms t and t' of $\mathcal{S}$, $[\![t]\!]_{\mathcal{D}} = [\![t']\!]_{\mathcal{D}}$ implies $t = t'$. In categorical terms, injectivity corresponds to faithfulness of the interpretation functor from $\mathcal{S}$ to $\mathcal{D}$. Injectivity is a natural and well studied question for denotational models of λ-calculi and term rewriting systems (see [11, 19]). In the framework of Linear Logic (LL, [12]) this question, addressed in [20], turned out to be remarkably complex: contrary to what happens in the λ-calculus, there exist semantics of LL proof-nets that are not injective, such as the coherent model which is injective only with respect to some fragments of LL (see [20]). After the first partial positive results obtained in [20], it took a long time to obtain some improvements: in [6], the injectivity of the relational model is proven for MELL (the multiplicative-exponential fragment of LL, sufficiently expressive to encode the λ-calculus) proof-structures that are connected, and eventually in [4] the first complete positive result is achieved, since the author proves that the relational model is injective for the all MELL proof-structures.

Ehrhard [7] introduced finiteness spaces, a denotational model of LL (and λ-calculus) which interprets formulas by topological vector spaces and proofs by analytical functions: in this model the operations of differentiation and the Taylor expansion make sense. Ehrhard and Regnier [8, 9, 10] internalized these operations in the syntax and thus introduced differential linear logic DiLL_0 (which encodes the resource λ-calculus, see [9]), where the promotion rule (the only one in LL which is responsible for introducing the !-modality and hence for creating resources available at will, marked by boxes in LL proof-structures) is replaced by three new "finitary" rules introducing !-modality which are perfectly symmetric to the rules for the ?-modality: this allows a more subtle analysis of the resources consumption during the cut-elimination process. At the syntactic level, the Taylor expansion decomposes a LL proof-structure in a (generally infinite) formal sum of DiLL_0 proof-structures, each of which contains resources usable only a fixed number of times. Roughly speaking, each element of the Taylor expansion $\mathcal{T}_R$ of a LL proof-structure R is a DiLL_0 proof-structure obtained from R by replacing each box B in R with n_B copies of its content (for any $n_B \in \mathbb{N}$), recursively.

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In the light of the differential approach, it is clear (and well-known) that the proof-structure of order 1 of the Taylor expansion of a λ-term (which is obtained by taking exactly one copy of the argument of each application) is enough to entirely determine the λ-term: if two λ-terms t_1 and t_2 have the same element of order 1 in their Taylor expansion, then $t_1 = t_2$. One can formulate the results of [6] and [4] by saying that, given two proof-structures R_1 and R_2, if there exists an appropriate DiLL_0 proof-structure, whose order *depends on* R_1 *and* R_2, which occurs in the Taylor expansions of both R_1 and R_2, then $R_1 = R_2$. We prove, in the present paper, for connected MELL, a result which is very much in the style of the one just mentioned for the λ-calculus: if two proof-structures R_1 and R_2 (with or without cuts) have the same elements of order 2 in their Taylor expansions (which is obtained by taking two copies of the content of each box), then $R_1 = R_2$ (the element of order 2 of the Taylor expansion of a connected MELL proof-structure is enough to entirely determine the proof-structure). Since it is known (see [13]) that the elements of the Taylor expansion of a λ-term/LL proof-structure is essentially an element of its interpretation in the relational model, we immediately obtain another proof of the injectivity of the relational model for connected MELL proof-structures.

It is widely acknowledged, in the LL community, that the subsystem of LL corresponding to the λ-calculus enjoys all the possible good properties, while many of them are lost in the general MELL fragment. Our result seems to suggest the following hierarchy:

1. full MELL, for which there does not seem to be a way to bound "a priori" the complexity of the element of the Taylor expansion allowing to distinguish two different proof-structures;
2. connected MELL (containing the λ-calculus) for which the element of order 2 of the Taylor expansion of a proof-structure is enough to entirely determine the proof-structure;
3. the λ-calculus, for which the element of order 1 of the Taylor expansion of a term is enough to entirely determine the term.

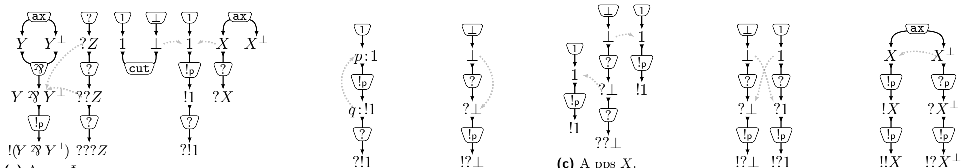
Outline After laying out precise definitions of proof-structure (§2) and Taylor expansion (§3), in §4 we show how a connected MELL proof-structure can be univocally computed by the point of order 2 of its Taylor expansion. Finally, in §5 we infer from this the injectivity of the relational model for connected MELL.

▶ **Notation.** We set $\mathcal{L}_{\mathsf{MELL}} = \{1, \bot, \otimes, ⅋, !, ?, ax, cut\}$. The set $\mathcal{F}_{\mathsf{MELL}}$ of MELL *formulas* is generated by the grammar: $A, B, C ::= X \mid X^\bot \mid 1 \mid \bot \mid A \otimes B \mid A ⅋ B \mid !A \mid ?A$, where X ranges over an infinite set of propositional variables. The linear negation is involutive, *i.e.* $A^{\bot\bot} = A$, and defined via De Morgan laws $1^\bot = \bot$, $(A \otimes B)^\bot = A^\bot ⅋ B^\bot$ and $(!A)^\bot = ?A^\bot$.

Let $\mathcal{A}$ be a set: $\mathscr{P}(\mathcal{A})$ is the power set of $\mathcal{A}$, $\bigcup \mathcal{A}$ is the union of $\mathcal{A}$, $\mathcal{A}^*$ is the set of finite sequences over $\mathcal{A}$. If $\mathcal{A}$ is ordered by $\leq$, for any $a \in \mathcal{A}$ we set $\downarrow_{\mathcal{A}} a = \{b \in \mathcal{A} \mid b \leq a\}$. The empty sequence is denoted by $()$. If $a = (a_1, \dots, a_n)$ with $n \in \mathbb{N}$, we set $|a| = n$ and, if $n > 0$, $a^- = (a_1, \dots, a_{n-1})$; if moreover $b = (b_1, \dots, b_m)$, we set $a \cdot b = (a_1, \dots, a_n, b_1, \dots, b_m)$. We write $a \sqsubseteq b$ if $a \cdot c = b$ for some finite sequence c. Let $\mathsf{f}\colon \mathcal{A} \to \mathcal{B}$ be a partial function: $\mathsf{dom}(\mathsf{f})$ and $\mathsf{im}(\mathsf{f})$ are the domain and image of f; the partial function $\bar{\mathsf{f}}\colon \mathscr{P}(\mathcal{A}) \to \mathscr{P}(\mathcal{B})$ is defined by $\bar{\mathsf{f}}(\mathcal{A}') = \{\mathsf{f}(a) \mid a \in \mathcal{A}' \cap \mathsf{dom}(\mathsf{f})\}$ for any $\mathcal{A}' \subseteq \mathcal{A}$.

2 A non-inductive syntax for proof structures

It is well-known that for linear logic proof-nets there is no "canonical" representation: every paper about them introduces its own syntax for proof-nets, and more generally for proof-structures, depending on the purposes of the paper. (Following [12], a proof-net is a proof-structure corresponding to a derivation in LL sequent calculus: proof-nets can be characterized among proof-structures via "geometric" correctness criteria, *e.g.* [2, 20]). The first aim of the syntax for proof-structures that we present here is to give a rigorous and compact definition of the following notions: (1) equality between proof-structures; (2) Taylor expansion of a proof-structure. The first point naturally leads us to adopt a low-level syntax with generalized ?- and !-links, similarly to [6]. Surprisingly enough, this choice can be made



(a) A pps $\varPhi$.
(b) Two pps, $\varPsi_1$ (on the left) and $\varPsi_2$ (on the right).
(c) A pps X.
(d) Two pps, $\varXi_1$ (on the left) and $\varXi_2$ (on the right).

Figure 1 Some examples of pps that are not DiLL-ps. See Def. 1 and 10.

compatible with the second point by giving a completely non-inductive definition of proof-structures, which is in keeping with the intuition that a proof-structure is a directed graph, plus further information about the borders of boxes. We have also taken care of minimizing the information required to identify a proof-structure, especially the borders of its boxes.

We use terminology of interactions nets [14, 9], even if properly speaking our objects are not interaction nets. So, for instance, our cells corresponds to links in [20]. Our syntax is inspired by [16, 17, 18, 21, 5, 6]. Our main technical novelties with respect to them are that:

- there are no wires (the same port may be auxiliary for some cell and principal for another cell), so axioms and cuts are cells, and our ports corresponds to edges in [20];
- boxes do not have an explicit constructor or cell, hence boxes and depth of a proof-structure are recovered in a non-inductive way.

As in [16, 17, 18] and unlike [5, 6], our syntactic objects are typed by MELL formulas: we have opted for a typed version only to keep out immediately the possibility of "vicious cycles" (see Fact 4). All the results in this paper can be adapted also to the untyped case.

Pre-proof-structures and isomorphisms We define here our basic syntactical object: *pre-proof-structure* (*pps* for short). All other syntactical objects, in particular proof-structures corresponding to the fragments or extensions of LL that we will consider (DiLL-, MELL- and DiLL_0-proof structures), are some special cases of pps. Essentially, a pps $\varPhi$ is a directed labelled graph $G_\varPhi$ called *ground-structure* (*gs* for short), plus a partial function $\mathsf{box}_\varPhi$ defined on certain edges (or nodes). The gs of $\varPhi$ represents a "linearised" proof-structure, *i.e.* $\varPhi$ without the border of its boxes; the partial function $\mathsf{box}_\varPhi$ marks the borders of the boxes of $\varPhi$. Examples of pps are in Fig. 1. Unlike [18, 6], our syntactical objects are not necessarily cut-free (nor with atomic axioms). Cut-elimination is not defined since it is not used here.

▶ **Definition 1** (Pre-proof-structure, ports, cells, ground-structure, fatness)**.** A *pre-proof-structure* (*pps* for short) is a 9-tuple $\varPhi = (\mathcal{P}_\varPhi, \mathcal{C}_\varPhi, \mathsf{tc}_\varPhi, \mathsf{P}^{\mathsf{pri}}_\varPhi, \mathsf{P}^{\mathsf{aux}}_\varPhi, \mathsf{P}^{\mathsf{left}}_\varPhi, \mathsf{tp}_\varPhi, \mathcal{C}^{\mathsf{box}}_\varPhi, \mathsf{box}_\varPhi)$ such that:

- $\mathcal{P}_\varPhi$ and $\mathcal{C}_\varPhi$ are finite sets, their elements are resp. the *ports* and the *cells* (or *links*) *of* $\varPhi$;
- $\mathsf{tc}_\varPhi$ is a function from $\mathcal{C}_\varPhi$ to $\mathcal{L}_{\mathsf{MELL}}$; for every $l \in \mathcal{C}_\varPhi$, $\mathsf{tc}_\varPhi(l)$ is the *label*, or *type*, *of* l; for every $t, t' \in \mathcal{L}_{\mathsf{MELL}}$, we write $l : t$ when $\mathsf{tc}_\varPhi(l) = t$, and we set $\mathcal{C}^t_\varPhi = \{l \in \mathcal{C}_\varPhi \mid l : t\}$ (whose elements are the t-*cells*, or t-*links*, *of* $\varPhi$) and $\mathcal{C}^{t,t'}_\varPhi = \mathcal{C}^t_\varPhi \cup \mathcal{C}^{t'}_\varPhi$;
- $\mathsf{P}^{\mathsf{pri}}_\varPhi$ is a function from $\mathcal{C}_\varPhi$ to $\mathscr{P}(\mathcal{P}_\varPhi)$ such that $\mathsf{im}(\mathsf{P}^{\mathsf{pri}}_\varPhi)$ covers $\mathcal{P}_\varPhi$ (that is, $\bigcup \mathsf{im}(\mathsf{P}^{\mathsf{pri}}_\varPhi) = \mathcal{P}_\varPhi$), and moreover, for all $l, l' \in \mathcal{C}_\varPhi$,
  - if $l \neq l'$ then $\mathsf{P}^{\mathsf{pri}}_\varPhi(l) \cap \mathsf{P}^{\mathsf{pri}}_\varPhi(l') = \emptyset$,
  - if $\mathsf{tc}_\varPhi(l) \in \{1, \bot, \otimes, \parr, !, ?\}$ then $\mathsf{card}(\mathsf{P}^{\mathsf{pri}}_\varPhi(l)) = 1$,
  - if $\mathsf{tc}_\varPhi(l) = ax$ (resp. $\mathsf{tc}_\varPhi(l) = cut$) then $\mathsf{card}(\mathsf{P}^{\mathsf{pri}}_\varPhi(l)) = 2$, (resp. $\mathsf{card}(\mathsf{P}^{\mathsf{pri}}_\varPhi(l)) = 0$);

  for any $l \in \mathcal{C}_\varPhi$, the elements of $\mathsf{P}^{\mathsf{pri}}_\varPhi(l)$ are the *principal ports*, or *conclusions*, *of* l *in* $\varPhi$;
- $\mathsf{P}^{\mathsf{aux}}_\varPhi$ is a function from $\mathcal{C}_\varPhi$ to $\mathscr{P}(\mathcal{P}_\varPhi)$ such that, for all $l, l' \in \mathcal{C}_\varPhi$,
  - if $l \neq l'$ then $\mathsf{P}^{\mathsf{aux}}_\varPhi(l) \cap \mathsf{P}^{\mathsf{aux}}_\varPhi(l') = \emptyset$,
  - if $\mathsf{tc}_\varPhi(l) \in \{1, \bot, ax\}$ then $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\varPhi(l)) = 0$; if $\mathsf{tc}_\varPhi(l) \in \{\otimes, \parr, cut\}$ then $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\varPhi(l)) = 2$;

for any $l \in \mathcal{C}_\Phi$, the elements of $\mathsf{P}^{\mathsf{aux}}_\Phi(l)$ are the *auxiliary ports*, or *premises*, *of* l *in* Φ;

- $\mathsf{P}^{\mathsf{left}}_\Phi \colon \mathcal{C}^{\otimes,\parr}_\Phi \to \mathcal{P}_\Phi$ is a function such that $\mathsf{P}^{\mathsf{left}}_\Phi(l) \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ for any $l \in \mathcal{C}^{\otimes,\parr}_\Phi$;
- $\mathsf{tp}_\Phi \colon \mathcal{P}_\Phi \to \mathcal{F}_{\mathsf{MELL}}$ is a function (we write $p : A$ and we say that A is the *type of* p, when $\mathsf{tp}_\Phi(p) = A$) such that, for any $l \in \mathcal{C}_\Phi$, one has
  - if $\mathsf{tc}_\Phi(l) = ax$ (resp. $\mathsf{tc}_\Phi(l) = cut$) and $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p_1, p_2\}$ (resp. $\mathsf{P}^{\mathsf{aux}}_\Phi(l) = \{p_1, p_2\}$), then $\mathsf{tp}_\Phi(p_1) = A$ and $\mathsf{tp}_\Phi(p_2) = A^\perp$, for some $A \in \mathcal{F}_{\mathsf{MELL}}$,
  - if $\mathsf{tc}_\Phi(l) = A \in \{1, \perp\}$ and $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\}$, then $\mathsf{tp}_\Phi(p) = A$,
  - if $\mathsf{tc}_\Phi(l) = \odot \in \{\otimes, \parr\}$, $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\}$, $\mathsf{P}^{\mathsf{aux}}_\Phi(l) = \{p_1, p_2\}$ and $\mathsf{P}^{\mathsf{left}}_\Phi(l) = p_1$, then $\mathsf{tp}_\Phi(p) = \mathsf{tp}_\Phi(p_1) \odot \mathsf{tp}_\Phi(p_2)$,
  - if $\mathsf{tc}_\Phi(l) = \lozenge \in \{!, ?\}$, $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\}$ and $\mathsf{P}^{\mathsf{aux}}_\Phi(l) = \{p_1, \dots, p_n\}$ ($n \in \mathbb{N}$), then $\mathsf{tp}_\Phi(p) = \lozenge A$ and $\mathsf{tp}_\Phi(p_i) = A$ for all $1 \leq i \leq n$, for some $A \in \mathcal{F}_{\mathsf{MELL}}$;
- $\mathcal{C}^{\mathsf{box}}_\Phi \subseteq \{l \in \mathcal{C}^!_\Phi \mid \mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l)) = 1\}$, the elements of $\mathcal{C}^{\mathsf{box}}_\Phi$ are the *box-cells of* Φ; for any $l \in \mathcal{C}^{\mathsf{box}}_\Phi$, its (unique) premise is denoted by $\mathsf{prid}_\Phi(l)$ and called the *principal door* or *pri-door of the box of* l (*in* R); we set $\mathcal{D}oors^!_\Phi = \bigcup \overline{\mathsf{P}^{\mathsf{aux}}_\Phi}(\mathcal{C}^{\mathsf{box}}_\Phi)$;[1]
- box_Φ is a partial function from $\bigcup \overline{\mathsf{P}^{\mathsf{aux}}_\Phi}(\mathcal{C}^{?,cut}_\Phi) \cup \mathcal{D}oors^!_\Phi$ to $\mathcal{C}^{\mathsf{box}}_\Phi$ such that $\mathsf{box}_\Phi(\mathsf{prid}_\Phi(l)) = l$ for all $l \in \mathcal{C}^{\mathsf{box}}_\Phi$.[2]

We set: $\mathcal{P}^{\mathsf{aux}}_\Phi = \bigcup \mathsf{im}(\mathsf{P}^{\mathsf{aux}}_\Phi)$, whose elements are the *auxiliary ports* of Φ; $\mathcal{P}^{\mathsf{free}}_\Phi = \mathcal{P}_\Phi \smallsetminus \mathcal{P}^{\mathsf{aux}}_\Phi$, whose elements are the *free ports* of Φ; and $\mathcal{C}^{\mathsf{free}}_\Phi = \{l \in \mathcal{C}_\Phi \mid \mathsf{P}^{\mathsf{pri}}_\Phi(l) \subseteq \mathcal{P}^{\mathsf{free}}_\Phi\}$, whose elements are the *free*, or *terminal*, *cells* of Φ.[3]

For any pps Φ, the *ground-structure* (*gs* for short) *of* Φ is the 7-tuple $G_\Phi = (\mathcal{P}_\Phi, \mathcal{C}_\Phi, \mathsf{tc}_\Phi, \mathsf{P}^{\mathsf{pri}}_\Phi, \mathsf{P}^{\mathsf{aux}}_\Phi, \mathsf{P}^{\mathsf{left}}_\Phi, \mathsf{tp}_\Phi)$.

A pps Φ is *fat* (resp. *strongly fat*) if $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l)) \geq 1$ (resp. $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l)) \geq 2$) for all $l \in \mathcal{C}^!_\Phi$.

Let us make some comments on Def. 1. Let Φ be a pps.

- The function $\mathsf{P}^{\mathsf{left}}_\Phi$ fixes an order on the two premises of any $\otimes$- and $\parr$-cell of Φ; the premises of the other types of cells are unordered, as well as the conclusions of the ax-cells.
- The conditions $\bigcup \mathsf{im}(\mathsf{P}^{\mathsf{pri}}_\Phi) = \mathcal{P}_\Phi$ and "for all $l, l' \in \mathcal{C}_\Phi$, if $l \neq l'$ then $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cap \mathsf{P}^{\mathsf{pri}}_\Phi(l') = \emptyset = \mathsf{P}^{\mathsf{aux}}_\Phi(l) \cap \mathsf{P}^{\mathsf{aux}}_\Phi(l')$" mean that every port is conclusion of exactly one cell and premise of at most one cell; the elements of $\mathcal{P}^{\mathsf{free}}_\Phi$ are the ports of Φ that are not premises of any cell.
- No condition is required for $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l))$ when $l \in \mathcal{C}^{!,?}_\Phi$: l can have $n \in \mathbb{N}$ premises since we use generalized ?- and !-cells for (co-)contraction, (co-)weakening and (co-)dereliction.
- The gs G_Φ of Φ is obtained from Φ by forgetting box_Φ and $\mathcal{C}^{\mathsf{box}}_\Phi$. In a way, G_Φ encodes the "geometric structure" of Φ (see below).

For any pps Φ, the fact that box_Φ is defined on $\mathcal{D}oors^!_\Phi$ simplifies the definition of the function $\mathsf{box}^{\mathsf{ext}}_\Phi$ (Def. 8), an extension of box_Φ that will be useful in the sequel. Any *box*-cell l of Φ is the starting point to compute the box associated with l: the ports in $\mathsf{box}^{-1}_\Phi(l)$ represent the border of this box. In general, not all !-cells of Φ with exactly one premise are *box*-cells.

► **Notation.** For any pps Φ we set $\mathcal{D}oors_\Phi = \mathsf{dom}(\mathsf{box}_\Phi)$ and $\mathcal{D}oors^?_\Phi = \mathcal{D}oors_\Phi \cap \bigcup \overline{\mathsf{P}^{\mathsf{aux}}_\Phi}(\mathcal{C}^?_\Phi)$, $\mathcal{D}oors^{cut}_\Phi = \mathcal{D}oors_\Phi \cap \bigcup \overline{\mathsf{P}^{\mathsf{aux}}_\Phi}(\mathcal{C}^{cut}_\Phi)$ and $\mathcal{C}^{\mathsf{bord}}_\Phi = \mathcal{C}^{\mathsf{box}}_\Phi \cup \{l \in \mathcal{C}^?_\Phi \mid \exists p \in \mathcal{D}oors^?_\Phi \cap \mathsf{P}^{\mathsf{aux}}_\Phi(l)\}$. From now on, $\bullet \notin \mathcal{C}_\Phi$ (in particular, $\bullet \notin \mathcal{C}^{\mathsf{box}}_\Phi$) for any pps Φ.

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[1] Hence, $\mathcal{D}oors^!_\Phi = \{\mathsf{prid}_\Phi(l) \mid l \in \mathcal{C}^{\mathsf{box}}_\Phi\}$, the set of premises of all *box*-cells of Φ.

[2] So, box_Φ is defined on $\mathcal{D}oors^!_\Phi$ and maps the (unique) premise of a *box*-cell l into l itself.

[3] Thus, a cell l of a pps Φ is in $\mathcal{C}^{\mathsf{free}}_\Phi$ iff either l is a ax-cell and both its conclusions are in $\mathcal{P}^{\mathsf{free}}_\Phi$, or l is a *cut*-cell, or l is neither an ax- nor a *cut*-cell and its unique conclusion is in $\mathcal{P}^{\mathsf{free}}_\Phi$.

(a)

$$\begin{tikzcd} \mathscr{P}(\mathcal{P}_\Phi) \arrow[d, "\overline{\varphi_\mathcal{P}}"'] & \mathcal{C}_\Phi \arrow[l, "\mathsf{P}^{\mathsf{aux}}_\Phi"] \arrow[r, "\mathsf{P}^{\mathsf{pri}}_\Phi"'] \arrow[d, "\varphi_\mathcal{C}"'] & \mathscr{P}(\mathcal{P}_\Phi) \arrow[d, "\overline{\varphi_\mathcal{P}}"] \\ \mathscr{P}(\mathcal{P}_\Psi) & \mathcal{C}_\Psi \arrow[l, "\mathsf{P}^{\mathsf{aux}}_\Psi"'] \arrow[r, "\mathsf{P}^{\mathsf{pri}}_\Psi"] & \mathscr{P}(\mathcal{P}_\Psi) \end{tikzcd} \quad \begin{tikzcd} \mathcal{C}_\Phi \arrow[r, "\mathsf{tc}_\Phi"'] \arrow[d, "\varphi_\mathcal{C}"'] & \mathcal{L}_{\mathsf{MELL}} \\ \mathcal{C}_\Psi \arrow[ur, "\mathsf{tc}_\Psi"'] & \end{tikzcd} \quad \begin{tikzcd} \mathcal{P}_\Phi \arrow[r, "\mathsf{tp}_\Phi"'] \arrow[d, "\varphi_\mathcal{P}"'] & \mathcal{F}_{\mathsf{MELL}} \\ \mathcal{P}_\Psi \arrow[ur, "\mathsf{tp}_\Psi"'] & \end{tikzcd} \quad \begin{tikzcd} \mathcal{C}^{\otimes,\parr}_\Phi \arrow[r, "\mathsf{P}^{\mathsf{left}}_\Phi"'] \arrow[d, "\varphi_\mathcal{C}"'] & \mathcal{P}_\Phi \arrow[d, "\varphi_\mathcal{P}"] \\ \mathcal{C}^{\otimes,\parr}_\Psi \arrow[r, "\mathsf{P}^{\mathsf{left}}_\Psi"] & \mathcal{P}_\Psi \end{tikzcd}$$

(b)

$$\begin{tikzcd} \mathcal{D}oors_\Phi \arrow[r, "\mathsf{box}_\Phi"'] \arrow[d, "\varphi_\mathcal{P}"] & \mathcal{C}^{\mathsf{box}}_\Phi \arrow[d, "\varphi_\mathcal{C}"'] \\ \mathcal{D}oors_\Psi \arrow[r, "\mathsf{box}_\Psi"] & \mathcal{C}^{\mathsf{box}}_\Psi \end{tikzcd}$$

Figure 2 Commutative diagrams for isomorphism of gs (Fig. 2a) and of pps (Fig. 2b). See Def. 5.

With any pps Φ are naturally associated a directed labelled graph $\mathfrak{G}(\Phi)$ whose nodes are the cells of Φ, labelled by their type; and whose oriented edges are the ports of Φ, labelled by their type; a premise (resp. conclusion) of a cell l is incoming in (resp. outgoing from) l.

Note that in the definition of $\mathfrak{G}(\Phi)$, $\mathcal{C}^{\mathsf{box}}_\Phi$ and box_Φ play no role, hence the gs G_Φ of Φ can naturally be seen as a labelled directed graph with a natural top-down orientation.

In the graphical representation of a pps Φ, a dotted arrow is depicted from a premise q of a ?-cell or *cut*-cell to the premise of a *box*-cell l as soon as $q \in \mathsf{box}_\Phi^{-1}(l)$. The label of a *box*-cell is marked as $!\mathsf{p}$. The names or types of ports and cells can be omitted.

▶ **Definition 2** ((Pre-)order on the ports of a pre-proof-structure)**.** Let Φ be a gs. The binary relation $<^1_\Phi$ on $\mathcal{P}_\Phi$ is defined by: $p <^1_\Phi q$ if there exists $l \in \mathcal{C}_\Phi$ such that $p \in \mathsf{P}^{\mathsf{pri}}_\Phi(l)$ and $q \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$. The preorder relation $\leq_\Phi$ on $\mathcal{P}_\Phi$ is the reflexive-transitive closure of $<^1_\Phi$. When $p \leq_\Phi q$ we say that q is *above* p. We write $p <_\Phi q$ if $p \leq_\Phi q$ and $p \neq q$.

In a pps Φ, the binary relation $\leq_\Phi$ has a geometric meaning (note that $\mathcal{C}^{\mathsf{box}}_\Phi$ and box_Φ, as well as tc_Φ, $\mathsf{P}^{\mathsf{left}}_\Phi$ and tp_Φ, play no role in Def. 2): for any $p, q \in \mathcal{P}_\Phi$, if $p \leq_\Phi q$ then in the directed graph $\mathfrak{G}(\Phi)$ there is a directed path from q to p that does not cross any *ax*-cell or *cut*-cell.

▶ **Remark 3** (Predecessor of a port)**.** Let Φ be a pps. For all $p \in \mathcal{P}^{\mathsf{aux}}_\Phi \smallsetminus \overline{\mathsf{P}^{\mathsf{aux}}_\Phi}(\mathcal{C}^{cut}_\Phi)$, there is a unique $q \in \mathcal{P}_\Phi$ (denoted by $\mathsf{pred}_\Phi(p)$, the *predecessor of* p) such that $q <^1_\Phi p$; moreover $\mathsf{pred}_\Phi(p) \neq p$. Indeed, by hypothesis p is a premise of some cell of Φ, but the only cells with more than one conclusion are the *ax*-cells, which have no premises; so, p is a premise of a cell of Φ having just one conclusion q; also, $\mathsf{tp}_G(q)$ is a proper subformula of $\mathsf{tp}_\Phi(p)$, thus $p \neq q$.

▶ **Fact 4** (Tree-like order on ports)**.** *Let Φ be a pps: $\leq_\Phi$ is a tree-like order relation on $\mathcal{P}_\Phi$.* Proof at p. 16

According to Fact 4, a pps Φ cannot have "vicious cycles" like for example a cell l such that $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cap \mathsf{P}^{\mathsf{aux}}_\Phi(l) \neq \emptyset$ (*i.e.* a port is both a premise and a conclusion of l).

The names of ports and cells of a pps (ports and cells being nothing but their names) will be important to define the labelled Taylor expansion (Def. 13), a more informative variant of the usual Taylor expansion (Def. 17). Nevertheless, a precise answer to the question "When two pps can be considered equal?" leads naturally to the notion of isomorphism between pps (Def. 5), inspired by the notion of isomorphism between graphs: intuitively, two pps are isomorphic if they are identical up to the names of their ports and cells.

▶ **Definition 5** (Isomorphism on ground-structures and pre-proof-structures)**.** Let Φ, Ψ be pps.

An *isomorphism from G_Φ to G_Ψ* is a pair $\varphi = (\varphi_\mathcal{P}, \varphi_\mathcal{C})$ of bijections $\varphi_\mathcal{P} \colon \mathcal{P}_\Phi \to \mathcal{P}_\Psi$ and $\varphi_\mathcal{C} \colon \mathcal{C}_\Phi \to \mathcal{C}_\Psi$ such that the diagrams in Fig. 2a commute. We write then $\varphi \colon G_\Phi \simeq G_\Psi$.

An *isomorphism from Φ to Ψ* is a pair $\varphi = (\varphi_\mathcal{P}, \varphi_\mathcal{C})$ of bijections $\varphi_\mathcal{P} \colon \mathcal{P}_\Phi \to \mathcal{P}_\Psi$ and $\varphi_\mathcal{C} \colon \mathcal{C}_\Phi \to \mathcal{C}_\Psi$ such that $\varphi \colon G_\Phi \simeq G_\Psi$, $\mathsf{im}(\varphi_\mathcal{C}{\restriction_{\mathcal{C}^{\mathsf{box}}_\Phi}}) = \mathcal{C}^{\mathsf{box}}_\Psi$, $\mathsf{im}(\varphi_\mathcal{P}{\restriction_{\mathcal{D}oors_\Phi}}) = \mathcal{D}oors_\Psi$ and the diagram in Fig. 2b commutes. We write then $\varphi \colon \Phi \simeq \Psi$.
If there is an isomorphism from Φ to Ψ, we say: *Φ and Ψ are isomorphic* and we write $\Phi \simeq \Psi$.

The relation $\simeq$ is an equivalence on the set of pps. Equivalence classes for $\simeq$ share the same graphical representation up to the order of the premises of their !- and ?-cells: any such a representation can be seen as a canonical representative of an equivalence class.

▶ **Remark 6.** Let Φ and Ψ be some pps with $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon G_\Phi \simeq G_\Psi$. We have:

1. $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l)) = \mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Psi(\varphi_{\mathcal{C}}(l)))$ for every $l \in \mathcal{C}_\Phi$, in particular Φ is fat (resp. strongly fat) iff Ψ is fat (resp. strongly fat); moreover, $\mathcal{P}^{\mathsf{free}}_\Psi = \overline{\varphi_{\mathcal{P}}}(\mathcal{P}^{\mathsf{free}}_\Phi)$ and $\mathcal{C}^{\mathsf{free}}_\Psi = \overline{\varphi_{\mathcal{C}}}(\mathcal{C}^{\mathsf{free}}_\Phi)$;
2. for every $p, q \in \mathcal{P}_\Phi$, $p \leq_\Phi q$ implies $\varphi_{\mathcal{P}}(p) \leq_\Psi \varphi_{\mathcal{P}}(q)$ ($\varphi_{\mathcal{P}}$ is non-decreasing).

DiLL-, DiLL_0- and MELL-proof-structures A pps Φ is a very "light" structure and in order to associate with any $l \in \mathcal{C}^{\mathsf{box}}_\Phi$ the sub-pps of Φ usually called the box of l, some conditions need to be satisfied: for example, boxes have to be ordered by a tree-like order (nesting), *cut*- and *ax*-cells cannot cross the border of a box, *etc.* We introduce here some restrictions to pps in order to define proof-structures corresponding to some fragments or extension of LL: MELL, DiLL and DiLL_0. Full differential linear logic (DiLL) is an extension of MELL (with the same language as MELL) provided with both promotion rule (*i.e.* boxes) and co-structural rules (the duals of the structural rules handling ?-modality) for the !-modality: DiLL_0 and MELL are particular subsystems of DiLL, respectively corresponding to the promotion-free (*i.e.* without boxes) fragment of DiLL and the fragment of DiLL without co-structural rules. Our interest for DiLL is just to have an unitary syntax subsuming MELL and DiLL_0: for this reason, unlike [17, 21], our DiLL-ps are not allowed to contain a set of DiLL-ps inside a box.

▶ **Definition 7** (DiLL_0-proof-structure)**.** A DiLL_0*-proof structure* (DiLL_0*-ps* or *diffnet* for short) is a pps Φ with $\mathcal{C}^{\mathsf{box}}_\Phi = \emptyset$. The set of DiLL_0-ps is denoted by $\mathbf{PS}_{\mathsf{DiLL}_0}$, and $\rho, \sigma, \ldots$ range over it.

So, a DiLL_0-ps ρ is a pps without *box*-cells: in this case, box_ρ is the empty function. Thus, any DiLL_0-ps ρ can be identified with its gs G_ρ.

To define the conditions that a pps has to fulfill to be a DiLL-ps, we first extend the partial function box_Φ to a function $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}$ that associates with every port p of Φ the "deepest" *box*-cell (if any) whose box contains p; it returns a dummy element $\bullet$ if p is not contained in any box.

▶ **Definition 8** (Extension of box_Φ)**.** Let Φ be a pps. The *extension of* box_Φ is a function $\mathsf{box}^{\mathsf{ext}}_\Phi \colon \mathcal{P}_\Phi \to \mathcal{C}^{\mathsf{box}}_\Phi \cup \{\bullet\}$ defined as follows: for any $p \in \mathcal{P}_\Phi$,

$$\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(p) = \begin{cases} \mathsf{box}_\Phi(\max_{\leq_\Phi}(\downarrow_{\mathcal{P}_\Phi} p \cap \mathcal{D}oors_\Phi)) & \text{if } \downarrow_{\mathcal{P}_\Phi} p \cap \mathcal{D}oors_\Phi \neq \emptyset \\ \bullet & \text{otherwise.} \end{cases}$$

For any pps Φ, the function $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}$ is well-defined since, for all $p \in \mathcal{P}_\Phi$, the set $\downarrow_{\mathcal{P}_\Phi} p \cap \mathcal{D}oors_\Phi$ is finite and totally ordered by $\leq_\Phi$, according to Fact 4: therefore the greatest element of $\downarrow_{\mathcal{P}_\Phi} p \cap \mathcal{D}oors_\Phi$ exists as soon as $\downarrow_{\mathcal{P}_\Phi} p \cap \mathcal{D}oors_\Phi \neq \emptyset$.

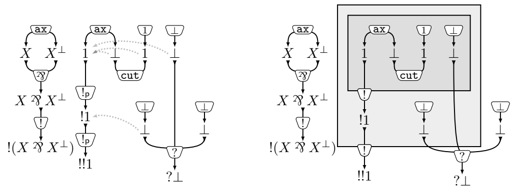
In a pps Φ, computing $\mathsf{box}^{\mathsf{ext}}_\Phi$ from box_Φ is simple. Given a port p of Φ, consider the maximal downwards path starting from p in the directed graph $\mathfrak{G}(G_\Phi)$: the first time the path bumps into a port $q \in \mathcal{D}oors_\Phi$ (if any), we set $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(p) = \mathsf{box}_\Phi(q)$; if the path does not bump into any $q \in \mathcal{D}oors_\Phi$, then $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(p) = \bullet$.

▶ **Definition 9** (Preorder on *box*-cells of a pre-proof-structure)**.** Let Φ be a pps. The binary relation $\leq_{\mathcal{C}^{\mathsf{box}}_\Phi}$ on $\mathcal{C}^{\mathsf{box}}_\Phi$ is defined by: $l \leq_{\mathcal{C}^{\mathsf{box}}_\Phi} l'$ (say l' *is above* l) iff there are $p, p' \in \mathcal{D}oors_\Phi$ such that $p \leq_\Phi p'$, $\mathsf{box}_\Phi(p) = l$ and $\mathsf{box}_\Phi(p') = l'$. We write $l <_{\mathcal{C}^{\mathsf{box}}_\Phi} l'$ if $l \leq_{\mathcal{C}^{\mathsf{box}}_\Phi} l'$ and $l \neq l'$.

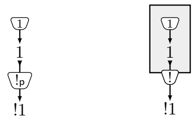
The binary relation $\leq_{\mathcal{C}^{\mathsf{box}}_\Phi \cup \{\bullet\}}$ on $\mathcal{C}^{\mathsf{box}}_\Phi \cup \{\bullet\}$ is defined by: $l \leq_{\mathcal{C}^{\mathsf{box}}_\Phi \cup \{\bullet\}} l'$ if either $l \leq_{\mathcal{C}^{\mathsf{box}}_\Phi} l'$ or $l = \bullet$. We write $l <_{\mathcal{C}^{\mathsf{box}}_\Phi \cup \{\bullet\}} l'$ when $l \leq_{\mathcal{C}^{\mathsf{box}}_\Phi \cup \{\bullet\}} l'$ and $l \neq l'$.

In any pps Φ, $\leq_{\mathcal{C}^{\mathsf{box}}_\Phi}$ is a preorder on $\mathcal{C}^{\mathsf{box}}_\Phi$, since $\leq_\Phi$ is a preorder on $\mathcal{P}_\Phi$. The preorder $\leq_{\mathcal{C}^{\mathsf{box}}_\Phi \cup \{\bullet\}}$ is the extension of $\leq_{\mathcal{C}^{\mathsf{box}}_\Phi}$ obtained by adding $\bullet$ as least element.

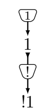
In Figure 1d, Ξ_1 is a pps such that $\leq_{\mathcal{C}^{\mathsf{box}}_{\Xi_1}}$ is not an order on $\mathcal{C}^{\mathsf{box}}_{\Xi_1}$; Ξ_2 is a pps such that $\leq_{\mathcal{C}^{\mathsf{box}}_{\Xi_2}}$ is an order but not a tree-like order on $\mathcal{C}^{\mathsf{box}}_{\Xi_2}$. A condition that a pps Φ must fulfill to be



(a) A DiLL-ps S which is neither a MELL-ps nor a DiLL_0-ps.



(b) A MELL-ps R_1 which is not a DiLL_0-ps.



(c) A DiLL_0-ps R_2 which is not a MELL-ps.

Figure 3 Some examples of DiLL-ps. In R_1 (Fig. 3b) $\mathcal{C}^{\mathsf{box}}_{R_1} = \{l\}$ and box_{R_1} is the empty function. In R_2 (Fig. 3c) $\mathcal{C}^{\mathsf{box}}_{R_2} = \emptyset$, so box_{R_2} is the empty function. Both S (Fig. 3a) and R_1 (Fig. 3b) are in two different presentations: the "arrow-like" one (on the left) and the "inductive-like" one (on the right).

a DiLL-ps is just that $\leq_{\mathcal{C}^{\mathsf{box}}_{\Phi}}$ is a tree-like order (or equivalently, $\leq_{\mathcal{C}^{\mathsf{box}}_{\Phi}\cup\{\bullet\}}$ is a rooted tree-like order whose root is $\bullet$): this amounts to the nesting of boxes (see [13] or Appendix B).

▶ **Definition 10** (DiLL-proof-structure and MELL-proof-structure)**.** A DiLL-*proof-structure* (DiLL-*ps* for short) is a pps Φ such that:

1. $\leq_{\mathcal{C}^{\mathsf{box}}_{\Phi}}$ is a tree-like order on $\mathcal{C}^{\mathsf{box}}_{\Phi}$;
2. $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(q)$ for all $l \in \mathcal{C}^{ax}_{\Phi}$ with $\mathsf{P}^{\mathsf{pri}}_{\Phi}(l) = \{p, q\}$ and all $l \in \mathcal{C}^{cut}_{\Phi}$ with $\mathsf{P}^{\mathsf{aux}}_{\Phi}(l) = \{p, q\}$;
3. for all $p \in \mathcal{D}oors^{!}_{\Phi} \cup \mathcal{D}oors^{?}_{\Phi}$, one has $\mathsf{box}_{\Phi}(p) \neq \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(\mathsf{pred}_{\Phi}(p))$;
4. for all $l \in \mathcal{C}^{\mathsf{box}}_{\Phi} \cup \{\bullet\}$ and $p \in \mathcal{D}oors^{!}_{\Phi}$, if $l <_{\mathcal{C}^{\mathsf{box}}_{\Phi}\cup\{\bullet\}} \mathsf{box}_{\Phi}(p)$ then $l \leq_{\mathcal{C}^{\mathsf{box}}_{\Phi}\cup\{\bullet\}} \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(\mathsf{pred}_{\Phi}(p))$.

A MELL-*proof-structure* (MELL-*ps* for short) is a DiLL-ps Φ such that $\mathcal{C}^{\mathsf{box}}_{\Phi} = \mathcal{C}^{!}_{\Phi}$. The set of DiLL-ps (resp. MELL-ps) is denoted by $\mathbf{PS}_{\mathsf{DiLL}}$ (resp. $\mathbf{PS}_{\mathsf{MELL}}$) and $R, S, \dots$ range over it.

In Def. 10, condition 2 means that a *cut*-cell (resp. *ax*-cell) cannot cross the border of a box, *i.e.* its premises (resp. conclusions) belong to the same boxes; the pps Φ in Fig. 1a does not fulfill condition 2. Condition 3 in Def. 10 entails that two ports on the border of the same box cannot be above each other (in the sense of $\leq_\Phi$); the pps Ψ_1 and Ψ_2 in Fig. 1b do not fulfill condition 3. Condition 4 in Def. 10 implies that the border of a box cannot have more than one !-cell; the pps X in Fig. 1c does not fulfill condition 4. See [13] for more details.

In [13] (or equivalently Appendix B) we show that the information encoded in a DiLL-ps R is enough to associate a box R_l with any *box*-cell l of R. So, as usual for LL, R_l can be graphically depicted (instead of using dotted arrows to pick out $\mathsf{box}_R^{-1}(l)$) by a rectangular frame containing all ports in $\mathsf{inbox}_R(l)$. Some examples of DiLL-ps are in Fig. 3.

▶ **Definition 11** (Content of the box, depth)**.** Let R be a DiLL-ps.

For any $l \in \mathcal{C}^{\mathsf{box}}_{R}$, the *content of the box of* l is $\mathsf{inbox}_R(l) = \{q \in \mathcal{P}_R \mid l \leq_{\mathcal{C}^{\mathsf{box}}_{R}} \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(q)\}$.

The function $\mathsf{box}^{\mathsf{ext}}_{\mathcal{C}_R} \colon \mathcal{C}_R \to \mathcal{C}^{\mathsf{box}}_{R}$ is defined by: for every $l \in \mathcal{C}_R \smallsetminus \mathcal{C}^{cut}_{R}$ (resp. $l \in \mathcal{C}^{cut}_{R}$), we set $\mathsf{box}^{\mathsf{ext}}_{\mathcal{C}_R}(l) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p)$ where $p \in \mathsf{P}^{\mathsf{pri}}_{R}(l)$ (resp. $p \in \mathsf{P}^{\mathsf{aux}}_{R}(l)$).[4]

For every $p \in \mathcal{P}_R$ and $l \in \mathcal{C}_R$, the *depths of* p *and* l *in* R are defined as follows: $\mathsf{depth}_R(p) = \mathsf{card}(\downarrow_{\mathcal{C}^{\mathsf{box}}_{R}}(\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p)))$ and $\mathsf{depth}_R(l) = \mathsf{card}(\downarrow_{\mathcal{C}^{\mathsf{box}}_{R}}(\mathsf{box}^{\mathsf{ext}}_{\mathcal{C}_R}(l)))$.

We set $\mathcal{P}^0_R = \{p \in \mathcal{P}_R \mid \mathsf{depth}_R(p) = 0\}$, $\mathcal{C}^0_R = \{l \in \mathcal{C}_R \mid \mathsf{depth}_R(l) = 0\}$ and $\mathcal{C}^{\mathsf{box}_0}_{R} = \mathcal{C}^{\mathsf{box}}_{R} \cap \mathcal{C}^0_R$. The *depth of* R is $\mathsf{depth}(R) = \sup\{\mathsf{depth}_R(p) \in \mathbb{N} \mid p \in \mathcal{P}_R\}$.

Given a DiLL-ps R, for any *box*-cell l in R, $\mathsf{inbox}_R(l)$ represents the set of ports contained in the box of l. According to Definition 11, the meaning of $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}$ is clear: for any port p of R, $\downarrow_{\mathcal{C}^{\mathsf{box}}_{R}}(\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p)) = \{l \in \mathcal{C}^{\mathsf{box}}_{R} \mid p \in \mathsf{inbox}_R(l)\}$ is the set of boxes in R containing p, and if $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \bullet$ then p has depth 0 (no box in R contains p), otherwise $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p)$ is the deepest

[4] For every $l \in \mathcal{C}_R$, $\mathsf{box}^{\mathsf{ext}}_{\mathcal{C}_R}(l)$ is well-defined by condition 2 in Def. 10. Note that, for any $l \in \mathcal{C}^{\mathsf{box}}_{R}$, $\mathsf{box}^{\mathsf{ext}}_{\mathcal{C}_R}(l)$ is the immediate predecessor of l in the tree-like order $\leq_{\mathcal{C}^{\mathsf{box}}_{R}\cup\{\bullet\}}$.

box-cell in R whose box contains p; the depth of p in R is the number of nested boxes in R containing p. According to Def. 11, for any *box*-cell l, $\mathsf{depth}_R(\mathsf{prid}_R(l)) = \mathsf{depth}_R(l) + 1$.

3 Computing the Taylor expansion of a DiLL-proof-structure

The Taylor expansion of a MELL-ps, or more generally a DiLL-ps, R is a (usually infinite) set of DiLL_0-ps: roughly speaking, each element of the Taylor expansion of R is obtained from R by replacing each box B in R with n copies of its content (for any $n \in \mathbb{N}$), recursively on the depth of R. Note that n depends not only on B but also on what "copy" of the contents of all boxes containing B we are considering. Usually, the Taylor expansion of MELL-ps [16, 18] is defined globally and inductively: with every MELL-ps R is directly associated its Taylor expansion (the whole set!) by induction on the depth of R. We adopt an alternative approach, which is pointwise and non-inductive: visually, it is exemplified by Fig. 4.

We introduce here *Taylor-functions*: a Taylor-function of a DiLL-ps R ascribes recursively a number of copies for each box of R. Any element of the Taylor expansion of R can be built from (at least) one element of the *proto-Taylor expansion* $\mathcal{T}_R^{\mathsf{proto}}$ of R, $\mathcal{T}_R^{\mathsf{proto}}$ being the set of Taylor-functions of R. We build in this way a more informative version of the Taylor expansion of R, the *labelled Taylor-expansion* $\mathcal{T}_R$ of R: one of the advantages of our pointwise and non-inductive approach is that it is easy to define the correspondence between ports and cells of any element ρ of the Taylor expansion of R and ports and cells of R (an operation intuitively clear but very awkward to define with the global and inductive approach), and to differentiate the various copies in ρ of the content of a same box in R. For this purpose, any port (or cell) of any DiLL_0-ps in the labelled Taylor expansion of R is of the shape (p, a), where p is the corresponding port (or cell) of R and the finite sequence a has to be intended as a list of indexes saying in which copy of the content of each box (p, a) is. These indexes are a syntactic counterpart of the ones used in the definition of k-experiment of *PLPS* in [6, Def. 35]. The information encoded in any element of the labelled Taylor expansion will be useful to prove some fundamental lemmas in §4. The usual Taylor expansion of a DiLL-ps R (whose elements do not contain this information, Def. 17) is then the quotient of $\mathcal{T}_R$ modulo isomorphism, *i.e.* modulo renaming of ports and cells of any DiLL_0-ps in $\mathcal{T}_R$.

▶ **Definition 12** (Taylor-function of a DiLL-proof-structure)**.** Let R be a DiLL-ps.

A *Taylor-function of* R is a function $\mathsf{f}\colon \mathcal{C}_R^{\mathsf{box}} \cup \{\bullet\} \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^*)$ such that:

1. *(depth compatibility)* $\mathsf{f}(\bullet) = \{(\,)\}$ and, for any $l \in \mathcal{C}_R^{\mathsf{box}}$ and $a \in \mathsf{f}(l)$, $|a| = \mathsf{depth}_R(\mathsf{prid}_R(l))$;
2. *(vertical downclosure)* for all $l, l' \in \mathcal{C}_R^{\mathsf{box}}$ such that $l \leq_{\mathcal{C}_R^{\mathsf{box}}} l'$, with $k = \mathsf{depth}_R(\mathsf{prid}_R(l))$ and $k' = \mathsf{depth}_R(\mathsf{prid}_R(l'))$ (so $k \leq k'$), if $(n_1, \dots, n_k, \dots, n_{k'}) \in \mathsf{f}(l')$ then $(n_1, \dots, n_k) \in \mathsf{f}(l)$.

The *proto-Taylor expansion of* R is the set $\mathcal{T}_R^{\mathsf{proto}}$ of Taylor-functions of R.

Note that the notion of Taylor-function of a DiLL-ps R relies only on the tree-like order on $\mathcal{C}_R^{\mathsf{box}}$, hence we could define the Taylor-function of any tree. By the vertical downclosure condition, any Taylor-function of a DiLL-ps R can be naturally presented as a tree-like order which is an "level-by-level expansion" of the tree-like order on $\mathcal{C}_R^{\mathsf{box}}$: see Fig. 4a-4c.

Our approach in defining the elements of the Taylor expansion of a DiLL-ps R separates the analysis of the number of copies to take for each (copy of) box of R (this information is given by any Taylor-function of R and is the most important one) from the operation of copying the content of each box (given by the function τ_R defined below). Indeed, with any Taylor-function of R one can easily associate an element of the (labelled) Taylor expansion of R (Def. 13).

▶ **Definition 13** (Labelled Taylor expansion)**.** Let R be a DiLL-ps.

The function $\tau_R \colon \mathcal{T}_R^{\mathsf{proto}} \to \mathbf{PS}_{\mathsf{DiLL}_0}$ associates with any $\mathsf{f} \in \mathcal{T}_R^{\mathsf{proto}}$ a DiLL_0-ps $\tau_R(\mathsf{f})$ defined by: $\mathcal{C}^{\mathsf{box}}_{\tau_R(\mathsf{f})} = \emptyset$, $\mathsf{box}_{\tau_R(\mathsf{f})}$ is the empty function, and

$$
\begin{aligned}
\mathcal{P}_{\tau_R(\mathsf{f})} &= \{(p,a) \mid p \in \mathcal{P}_R \text{ and } a \in \mathsf{f}(\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p))\} \\
\mathcal{C}_{\tau_R(\mathsf{f})} &= \{(l,a) \mid l \in \mathcal{C}_R \text{ and } a \in \mathsf{f}(\mathsf{box}^{\mathsf{ext}}_{\mathcal{C}_R}(l))\} \\
\mathsf{tc}_{\tau_R(\mathsf{f})}((l,a)) &= \mathsf{tc}_R(l) \qquad \text{for every } (l,a) \in \mathcal{C}_{\tau_R(\mathsf{f})} \\
\mathsf{P}^{\mathsf{pri}}_{\tau_R(\mathsf{f})}((l,a)) &= \{(p,a) \mid p \in \mathsf{P}^{\mathsf{pri}}_R(l)\} \qquad \text{for every } (l,a) \in \mathcal{C}_{\tau_R(\mathsf{f})} \\
\mathsf{P}^{\mathsf{aux}}_{\tau_R(\mathsf{f})}((l,a)) &= \{(p,b) \mid p \in \mathsf{P}^{\mathsf{aux}}_R(l),\ a \sqsubseteq b \in \mathsf{f}(\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p))\} \quad \text{for any } (l,a) \in \mathcal{C}_{\tau_R(\mathsf{f})} \\
\mathsf{P}^{\mathsf{left}}_{\tau_R(\mathsf{f})}((l,a)) &= (\mathsf{P}^{\mathsf{left}}_R(l), a) \qquad \text{for every } (l,a) \in \mathcal{C}^{\otimes,\parr}_{\tau_R(\mathsf{f})} \\
\mathsf{tp}_{\tau_R(\mathsf{f})}((p,a)) &= \mathsf{tp}_R(p) \qquad \text{for every } (p,a) \in \mathcal{P}_{\tau_R(\mathsf{f})}
\end{aligned}
$$

The *labelled Taylor expansion of* R is the set of DiLL_0-ps $\mathcal{T}_R = \mathsf{im}(\tau_R)$.

The proof that $\tau_R(\mathsf{f})$ is a DiLL_0-ps for any DiLL-ps R and any Taylor-function f of R, is left to the reader. The set $\mathcal{T}_R$ (as well as $\mathcal{T}_R^{\mathsf{proto}}$) is infinite iff $\mathsf{depth}(R) > 0$.

Note that when $l \in \mathcal{C}^{\mathsf{bord}}_R$, the condition $a \sqsubseteq b$ when defining $\mathsf{P}^{\mathsf{aux}}_{\tau_R(\mathsf{f})}((l,a))$ in Def. 13 plays a crucial role: for instance, given the MELL-ps R as in Fig. 4a and the Taylor-function f of R as in Fig. 4c, the premises of the !-cell $(l_1,(1))$ of $\tau_R(\mathsf{f})$ (whose conclusion is $(r_1,(1))$ in Fig. 4d) are $(p_1,(1,1))$, $(p_1,(1,2))$, $(p_1,(1,3))$, and not $(p_1,(2,1))$, since $(1) \not\sqsubseteq (2,1)$.

▶ **Remark 14** (Canonicity)**.** Given a DiLL-ps R and $\mathsf{f} \in \mathcal{T}_R^{\mathsf{proto}}$, we say that f is *canonical* if

- *(horizontal downclosure)* for every $l \in \mathcal{C}^{\mathsf{box}}_R$, if $(n_1,\dots,n_m) \in \mathsf{f}(l)$ then $n_1,\dots,n_m \in \mathbb{N}^+$ and $(n_1,\dots,n_{m-1},k) \in \mathsf{f}(l)$ for any $1 \leq k \leq n_m$.

A $\rho \in \mathcal{T}_R$ is *canonical* if $\rho = \tau_R(\mathsf{f})$ for some canonical $\mathsf{f} \in \mathcal{T}_R^{\mathsf{proto}}$. In any canonical DiLL_0-ps of $\mathcal{T}_R$ the various copies of the content of a box are numbered sequentially. It can easily be shown that for any $\rho \in \mathcal{T}_R$, there is a canonical $\sigma \in \mathcal{T}_R$ such that $\rho \simeq \sigma$.

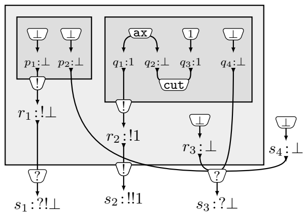
The next example shows how to compute an element ρ of the labelled Taylor expansion of a DiLL-ps R starting from R and a Taylor-function of R. It shows also the information encoded in ρ with respect to R: the correspondence between ports (and cells) of ρ and ports (and cells) of R, and the differentiation of the various copies in ρ of the content of a same box in R.

▶ **Example 15.** Let R be the MELL-ps as in Fig. 4a (the tree-like order on $\mathcal{C}^{\mathsf{box}}_R$ is in Fig. 4b) and f be the Taylor-function of R as in Fig. 4c. The DiLL_0-ps $\tau_R(\mathsf{f}) \in \mathcal{T}_R$ obtained from f by applying Def. 13 is in Fig. 4d. Note that the ports $(p_2,(1,2))$ and $(p_2,(2,1))$ are two ports of $\tau_R(\mathsf{f})$ corresponding to the port p_2 of R: more precisely, $(p_2,(1,2))$ (resp. $(p_2,(2,1))$) is in the second (resp. first) copy of the content of the box of l_1 which is in the first (resp. second) copy of the content of o. Analogously for the other ports and cells of $\tau_R(\mathsf{f})$.

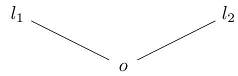
▶ **Definition 16** (Forgetful functions)**.** Let $R \in \mathbf{PS}_{\mathsf{DiLL}}$ and $\rho \in \mathcal{T}_R$. The *forgetful functions* $\mathsf{forget}^{\rho,R}_{\mathcal{P}} \colon \mathcal{P}_\rho \to \mathcal{P}_R$ and $\mathsf{forget}^{\rho,R}_{\mathcal{C}} \colon \mathcal{C}_\rho \to \mathcal{C}_R$ are defined by: $\mathsf{forget}^{\rho,R}_{\mathcal{P}}((p,a)) = p$ and $\mathsf{forget}^{\rho,R}_{\mathcal{C}}((l,b)) = l$ for all $(p,a) \in \mathcal{P}_\rho$ and $(l,b) \in \mathcal{C}_\rho$.

By forgetting the indexes associated with the ports and cells of $\rho \in \mathcal{T}_R$, the functions $\mathsf{forget}^{\rho,R}_{\mathcal{P}}$ and $\mathsf{forget}^{\rho,R}_{\mathcal{C}}$ make explicit the correspondence (neither injective nor surjective) between ports and cells of ρ and ports and cells of R, implicitly given in Def. 13.

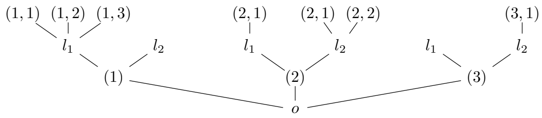
It is easy to show that, even if the function τ_R for any DiLL-ps R is injective, there may exist two different Taylor-functions of R whose images via τ_R are different but isomorphic: the labelled Taylor expansion of a DiLL-ps may contain several elements which are isomorphic



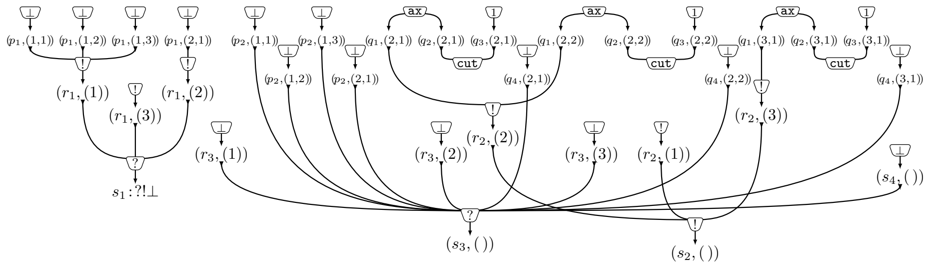
(a) A MELL-ps R, where o (resp. l_1;l_2) is the *box*-cell whose conclusion is of type !!1 (resp. !⊥; !1).



(b) The tree-like order on $\mathcal{C}_R^{\mathsf{box}}$.



(c) The tree-like presentation of a Taylor-function f of R defined by: $\mathsf{f}(o) = \{(1),(2),(3)\}$, $\mathsf{f}(l_1) = \{(1,1),(1,2),(1,3),(2,1)\}$ and $\mathsf{f}(l_2) = \{(2,1),(2,2),(3,1)\}$.



(d) A DiLL_0-ps $\tau_R(\mathsf{f}) \in \mathcal{T}_R$ (types of the ports are omitted).

Figure 4 From a MELL-ps R (Figure 4a) to an element of the labelled Taylor expansion of R (Figure 4d), via a Taylor-function of R (Figure 4c). See also Example 15.

and differ from each other only by the name of their ports and cells. Moreover, the Taylor expansion is not closed by isomorphism: from $\rho \in \mathcal{T}_R$ for some DiLL-ps R and $\sigma \simeq \rho$, it does not follow that $\sigma \in \mathcal{T}_R$ (and there might even exist a DiLL-ps $S \not\simeq R$ with $\sigma \in \mathcal{T}_S$). This means that, although ρ and σ are isomorphic as DiLL_0-ps, all the information about R available in ρ thanks to the names of its ports and cells is lost in σ.

The definition of Taylor expansion of a MELL-ps coming from [10] and used in [16, Def. 9] and [18, Def. 5] forgets all the information encoded in our labelled Taylor expansion.

▶ **Definition 17** (Taylor expansion of a DiLL-proof-structure)**.** Let R be a DiLL-ps. The *Taylor expansion of* R is $\mathcal{T}_R^{\simeq} = \big\{\{\tau \in \mathbf{PS}_{\mathsf{DiLL}_0} \mid \tau \simeq \rho\} \mid \rho \in \mathcal{T}_R\big\}$.

Let R be a DiLL-ps: the binary relation $\approx_R$ on $\mathbf{PS}_{\mathsf{DiLL}_0}$ defined by "$\tau \approx_R \tau'$ iff there is $\rho \in \mathcal{T}_R$ such that $\tau \simeq \rho \simeq \tau'$" is a partial equivalence relation, and, for any $\rho \in \mathcal{T}_R$, $\{\tau \in \mathbf{PS}_{\mathsf{DiLL}_0} \mid \tau \simeq \rho\}$ is a partial equivalence class on $\mathbf{PS}_{\mathsf{DiLL}_0}$ modulo $\approx_R$. Morally, $\mathcal{T}_R^{\simeq}$ is the quotient of $\mathcal{T}_R$ modulo isomorphism, *i.e.* modulo renaming of ports and cells of each element of $\mathcal{T}_R$: any element of $\mathcal{T}_R^{\simeq}$ can be seen as an element of $\mathcal{T}_R$ where all the information encoded in the names of its ports and cells is forgotten. Clearly, if $R \simeq S$ then $\mathcal{T}_R^{\simeq} = \mathcal{T}_S^{\simeq}$.

Let us stress the differences between $\mathcal{T}_R$ and $\mathcal{T}_R^{\simeq}$ of a DiLL-ps R. Given a (co-)contraction cell l of $\rho \in \mathcal{T}_R$ (*i.e.* $l \in \mathcal{C}_\rho^{!,?}$ and $\mathsf{card}(\mathsf{P}_\rho^{\mathsf{aux}}(l)) \geq 2$), it is possible to distinguish if l is a "real" (co-)contraction (*i.e.* the corresponding !- or ?-cell l' of R has at least 2 premises) or not (and then l' is in the border of some box and has only one premise which is in $\mathcal{D}oors_R^! \cup \mathcal{D}oors_R^?$): only in the first case there are two premises (p,a) and (q,b) of l with $p \neq q$. We can make this distinction via the information encoded in the names of ports and cells of $\rho \in \mathcal{T}_R$, but in general we are not able to do that in (any representative of) an element of $\mathcal{T}_R^{\simeq}$.

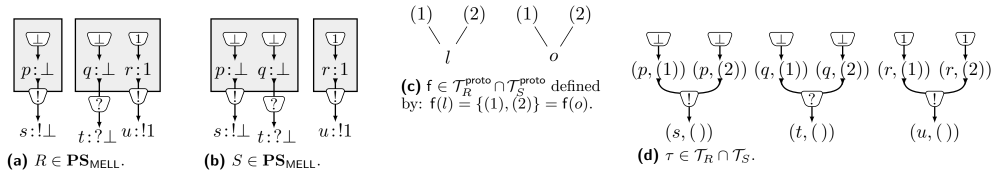


Figure 5 Two non-isomorphic MELL-ps R (Fig. 5a) and S (Fig. 5b), where $\mathcal{C}_R^{\mathsf{box}} = \{l, o\} = \mathcal{C}_S^{\mathsf{box}}$. The DiLL_0-ps $\tau \in \mathcal{T}_R \cap \mathcal{T}_S$ (Fig. 5d) is generated by the Taylor-function f of R and S (Fig. 5c).

Nevertheless, the information encoded in the labelled Taylor expansion of a DiLL-ps has some limitations: in general, a DiLL-ps R is not completely characterized by any $\rho \in \mathcal{T}_R$ (even if ρ is R-fat or strongly R-fat, see Def. 18 below), *i.e.* the fact that $\rho \in \mathcal{T}_R \cap \mathcal{T}_S$ for some DiLL-ps R and S does not imply $R \simeq S$. For instance, the DiLL_0-ps τ in Fig. 5d is an element of both $\mathcal{T}_R$ and $\mathcal{T}_S$, where R and S are as in Fig. 5a and 5b, respectively.

Elements of special interest of the labelled Taylor expansion of a DiLL proof-structure

▶ **Definition 18** (R-fatness, k-diffnet of a DiLL-ps)**.** Let $R \in \mathbf{PS}_{\mathsf{DiLL}}$, $\rho \in \mathcal{T}_R$ and $k \in \mathbb{N}$.

- ρ is *R-fat* (resp. *strongly R-fat*) if, for every $(l, b) \in \mathcal{C}_\rho^!$ such that $l \in \mathcal{C}_R^{\mathsf{box}}$, one has $\mathsf{card}(\mathsf{P}_\rho^{\mathsf{aux}}((l, b))) \geq 1$ (resp. $\mathsf{card}(\mathsf{P}_\rho^{\mathsf{aux}}((l, b))) \geq 2$);
- ρ is a *k-diffnet of R* if $\mathsf{card}(\mathsf{P}_\rho^{\mathsf{aux}}((l, b))) = k$ for any $(l, b) \in \mathcal{C}_\rho^!$ such that $l \in \mathcal{C}_R^{\mathsf{box}}$.

Given a DiLL-ps R and $\rho \in \mathcal{T}_R$: ρ is R-fat (resp. strongly R-fat) when ρ is obtained by taking at least one (resp. two) copies of the content of any box in R; ρ is a k-diffnet of R when ρ is obtained by taking exactly k copies of the content of any box in R. Any k-diffnet of R with $k \geq 1$ (resp. $k \geq 2$) is R-fat (resp. strongly R-fat). Given $k \in \mathbb{N}$, all k-diffnets of R are isomorphic, and there is a unique canonical k-diffnet of R. Following [6, Def. 16], it can be shown that the LPS of R is univocally determined by any R-fat $\rho \in \mathcal{T}_R$.

▶ **Fact 19** (Isomorphism of gs)**.** *Let $R, S \in \mathbf{PS}_{\mathsf{DiLL}}$ and ρ (resp. σ) be a 1-diffnet of R (resp. S).* Proof at p. 18

1. *The functions $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ and $\mathsf{forget}_{\mathcal{C}}^{\rho,R}$ are bijective, and $(\mathsf{forget}_{\mathcal{P}}^{\rho,R}, \mathsf{forget}_{\mathcal{C}}^{\rho,R}) \colon G_\rho \simeq G_R$.*
2. *Suppose $\varphi_1 \colon \rho \simeq \sigma$. Let $\varphi_{\mathcal{P}} \colon \mathcal{P}_R \to \mathcal{P}_S$ and $\varphi_{\mathcal{C}} \colon \mathcal{C}_R \to \mathcal{C}_S$ be functions defined by (for all $p \in \mathcal{P}_R$, $l \in \mathcal{C}_R$ and $a, b \in \mathbb{N}^*$ with $(p, a) \in \mathcal{P}_\rho$ and $(l, b) \in \mathcal{C}_\rho$): $\varphi_{\mathcal{P}}(p) = \mathsf{forget}_{\mathcal{P}}^{\sigma,S}(\varphi_{1\mathcal{P}}((p, a)))$ and $\varphi_{\mathcal{C}}(l) = \mathsf{forget}_{\mathcal{C}}^{\sigma,S}(\varphi_{1\mathcal{C}}((l, b)))$. Then, $\varphi_{\mathcal{P}}$ and $\varphi_{\mathcal{C}}$ are bijective and $(\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon G_R \simeq G_S$.*

The fact that $\rho \in \mathcal{T}_R$ for some DiLL-ps R and $\sigma \simeq \rho$ do not imply that $\sigma \in \mathcal{T}_R$ (and there may exist a DiLL-ps $S \not\simeq R$ such that $\sigma \in \mathcal{T}_S$), so all the information about R available in ρ thanks to the names of its ports and cells is lost in σ, though ρ and σ "morally" represent the same object: in general looking at σ one is not able to recognize where the border of the boxes in R are. Fact 19.2 only says that if R, S are DiLL-ps and ρ (resp. σ) is the 1-diffnet of R (resp. S) with $\varphi_1 \colon \rho \simeq \sigma$, then φ_1 induces an isomorphism φ from the gs G_R of R to the gs G_S of S, but in general φ does not make diagram in Fig. 2b (Def. 5) commute. This is not surprising, since a 1-diffnet of a DiLL-ps R is essentially the gs of R (Fact 19.1), *i.e.* R having forgotten the border of boxes in R.

4 Connected case: computing a MELL-ps from its Taylor expansion

We show here our main result (Thm. 26): a connected (in the sense of Def. 22) MELL-ps R is completely characterized by any $\gamma \in \mathcal{T}_R^{\simeq}$ strongly fat (according to Def. 1, (strong) fatness is not defined for a set of pps, but this notion can be extended to a set of isomorphic pps thanks to Remark 6.1). The idea is that, by means of the "geometry" of γ (the same in all elements of γ, since they are isomorphic), we can recover the information about R encoded in the names of ports and cells of some *suitable* $\rho \in \mathcal{T}_R \cap \gamma$: in particular, we can identify the "real" contraction cells from the "fake" ones. A key-tool for this approach is the notion of ?-accessibility (Def. 20): it allows to separate the different copies of the content of a box, so it plays at a syntactic level the same role played by bridges in [6, Def. 73]. Intuitively, q is a ?-accessible port from p if there is a path in $\mathfrak{G}(\Phi)$ seen as undirected graph (see page 5) starting upward (since $p_0 \neq p_n$ in rules (iii)-(iv) of Def. 20) from p and ending in q, paying attention that, when crossing downward a cell l with type ? (here "upward" and "downward" are in the sense of the order relation $\leq_\Phi$ of Def. 2), we require that all the premises of l are reachable by a path starting upward from p.

▶ **Definition 20** (?-path, ?-accessibility)**.** Let Φ be a pps. A *?-path on Φ (from p_0 to p_n)* is a finite sequence $(p_0, \dots, p_n)$ of ports of Φ defined by induction as follows:

(i) (p) is a ?-path for any $p \in \mathcal{P}_\Phi$;
(ii) if $\vec{p} = (p_0, \dots, p_n)$ is a ?-path where $p_n \in \mathsf{P}^{\mathsf{pri}}_\Phi(l)$ for some $l \in \mathcal{C}_\Phi$, then $\vec{p}\cdot q$ is a ?-path, for any $q \in (\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cup \mathsf{P}^{\mathsf{aux}}_\Phi(l)) \smallsetminus \{p_n\}$;
(iii) if $\vec{p} = (p_0, \dots, p_n)$ is a ?-path with $p_n \in \mathsf{P}^{\mathsf{aux}}_\Phi(l) \smallsetminus \{p_0\}$ for some $l \in \mathcal{C}_\Phi$ such that $\mathsf{tc}_\Phi(l) \neq ?$, then $\vec{p}\cdot q$ is a ?-path, for any $q \in (\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cup \mathsf{P}^{\mathsf{aux}}_\Phi(l)) \smallsetminus \{p_n\}$;
(iv) if $\vec{p} = (p_0, \dots, p_n)$ is a ?-path with $p_n \in \mathsf{P}^{\mathsf{aux}}_\Phi(l) \smallsetminus \{p_0\}$ for some $l \in \mathcal{C}^?_\Phi$, if for any $r \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ there is a ?-path from p_0 to r, then $\vec{p}\cdot q$ is a ?-path, for any $q \in (\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cup \mathsf{P}^{\mathsf{aux}}_\Phi(l)) \smallsetminus \{p_n\}$.

For every $p \in \mathcal{P}_\Phi$, the set of the *?-accessible ports from p in Φ* is defined as $\mathsf{acces}^?_\Phi(p) := \{q \in \mathcal{P}_\Phi \mid \text{there is a ?-path in } \Phi \text{ from } p \text{ to } q\}$.

According to Def. 20, given a pps Φ and $p \in \mathcal{P}_\Phi$, the set of ?-accessible ports from p in Φ

- is upward-closed (rule (ii)): if $q \in \mathsf{acces}^?_\Phi(p)$ and $q \leq_R q'$ then $q' \in \mathsf{acces}^?_\Phi(p)$;
- is "often" downward-closed (rules (iii)-(iv)): if $q \in \mathsf{acces}^?_\Phi(p)$ and $q' \notin \mathsf{acces}^?_\Phi(p)$ with $q \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ and $q' \in \mathsf{P}^{\mathsf{pri}}_\Phi(l)$ for some $l \in \mathcal{C}_\Phi$, then $p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$, or $l \in \mathcal{C}^?_\Phi$ and $\mathsf{P}^{\mathsf{aux}}_\Phi(l) \not\subseteq \mathsf{acces}^?_\Phi(l)$;
- crosses *ax*-cells and *cut*-cells (rules (ii)-(iii)): if $l \in \mathcal{C}^{ax}_\Phi$ then either $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \subseteq \mathsf{acces}^?_\Phi(p)$ or $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cap \mathsf{acces}^?_\Phi(p) = \emptyset$; if $l \in \mathcal{C}^{cut}_\Phi$ then either $\mathsf{P}^{\mathsf{aux}}_\Phi(l) \subseteq \mathsf{acces}^?_\Phi(p)$ or $\mathsf{P}^{\mathsf{aux}}_\Phi(l) \cap \mathsf{acces}^?_\Phi(p) = \emptyset$.

The set $\mathcal{C}^{\mathsf{box}}_\Phi$ and the partial function box_Φ play no role in Def. 20: in other words, ?-paths and ?-accessibility can be equivalently defined in the gs G_Φ of Φ.

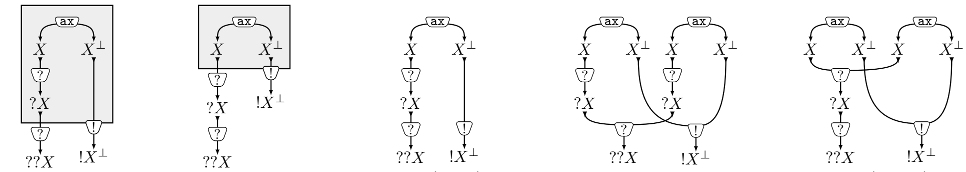
▶ **Remark 21.** Recalling Remark 6.2, one can easily see that, if Φ and Ψ are pps such that $\varphi\colon G_\Phi \simeq G_\Psi$, then for every $p \in \mathcal{P}_\Phi$ one has: $\overline{\varphi_\mathcal{P}}(\mathsf{acces}^?_\Phi(p)) = \mathsf{acces}^?_\Psi(\varphi_\mathcal{P}(p))$.

We now define the geometric key-notion of box-connectedness: a DiLL-ps is box-connected if, seen as an undirected graph, what is *inside* any box is recursively connected, that is (following [20, 6]), for any two ports p and q on the border of a same box, p and q are connected by a path crossing only ports with depth at least the depth of p (and q). Formally, our definition relies instead on ?-paths, which are a tool used in the proof of Lemma 23.

▶ **Definition 22** (?-path inside a box, box-connectedness)**.** Given $R \in \mathbf{PS}_{\mathsf{DiLL}}$ and $l \in \mathcal{C}^{\mathsf{box}}_R$, a ?-path $\vec{p} = (p_0, \dots, p_n)$ in R is *inside the box of l* if $p_i \in \mathsf{inbox}_R(l)$ for all $0 \leq i \leq n$.

A DiLL-ps R is *box-connected* if, for any $l \in \mathcal{C}^{\mathsf{box}}_R$ and $p \in \mathsf{inbox}_R(l)$, there is a ?-path in R from $\mathsf{prid}_R(l)$ to p inside the box of l.

For example, the DiLL-ps R_1 and R_2 in Fig. 3b-3c, and R and S in Fig. 6a-6b are box-connected; the DiLL-ps R and S in Fig. 4a and 3a are not box-connected. Clearly, any DiLL_0-ps, or more generally, any DiLL-ps R such that $\mathcal{D}oors^?_R = \emptyset = \mathcal{D}oors^{cut}_R$, is box-connected.



(a) $R \in \mathbf{PS}_{\mathsf{MELL}}$ (b) $S \in \mathbf{PS}_{\mathsf{MELL}}$ (c) $\tau_1 \in \mathcal{T}_R^{\simeq} \cap \mathcal{T}_S^{\simeq}$ (d) $\rho_2 \in \mathcal{T}_R^{\simeq} \smallsetminus \mathcal{T}_S^{\simeq}$ (e) $\sigma_2 \in \mathcal{T}_S^{\simeq} \smallsetminus \mathcal{T}_R^{\simeq}$

Figure 6 Two non-isomorphic box-connected MELL-ps R (Fig. 6a) and S (Fig. 6b), having in their Taylor expansions the same element τ_1 of order 1 (*i.e.* the set of DiLL_0-ps isomorphic to a 1-diffnet of R, Fig. 6c), but two different elements ρ_2 (Fig. 6d) and σ_2 (Fig. 6e) of order 2, respectively.

We stress that the box-connectedness condition (a crucial hypothesis in our main result) is quite general and not *ad hoc*. Indeed, it can be proven that: any ACC[5] DiLL-ps (in particular, any MELL-ps coming from a derivation in MELL sequent calculus without mix-rule) having neither $\bot$-cells nor weakenings (*i.e.* ?-cells with no premises) inside boxes is box-connected; any MELL-ps which is the translation of a λ-term (according to the call-by-name type identity $o = {!o} \multimap o$) is box-connected; box-connectedness is preserved under cut-elimination.

Box-connection and Taylor expansion Given a box-connected DiLL-ps R and a strongly R-fat $\rho \in \mathcal{T}_R$, all information encoded in the indexes of ports and cells of ρ can be recovered in a "geometric" way via ?-accessibility, without looking at the names of ports and cells of ρ: by Lemma 23, in ρ the copy with index a of the content of the box associated with a *box*-cell l of R is exactly the set of ?-accessible ports from the premise $(\mathsf{prid}_R(l), a)$ of the !-cell (l, a^-) of ρ.

▶ **Lemma 23** (Geometric characterization of the copies of a box in an element of the labelled Taylor expansion). *Let R be a DiLL-ps, $\rho \in \mathcal{T}_R$ and $(p, a) \in \mathcal{P}_\rho$ with $p = \mathsf{prid}_R(l)$ for some $l \in \mathcal{C}_R^{\mathsf{box}}$.*[6] *Let $\mathcal{P}_\rho^{l,a} = \{(q, a{\cdot}b) \in \mathcal{P}_\rho \mid b \in \mathbb{N}^* \text{ and } q \in \mathsf{inbox}_R(l)\}$. If R is box-connected and ρ is strongly R-fat, then $\mathcal{P}_\rho^{l,a} = \mathsf{acces}_\rho^?((p, a))$ and thus $\mathsf{inbox}_R(l) = \overline{\mathsf{forget}_{\mathcal{P}}^{\rho,R}}(\mathsf{acces}_\rho^?((p, a)))$.* Proof at p. 20

In the proof of Lemma 23, the hypothesis of box-connectedness (resp. strong R-fatness) ensures that the ?-accessible ports from $(\mathsf{prid}_R(l), a)$ in ρ contain at least (resp. at most) all the content of the copy with index a of the content of the box associated with the *box*-cell l of R. In Fig. 5, τ is a 2-diffnet of both R and S (so is strongly R- and S-fat) but R and S are not box-connected, and indeed the ?-accessible ports from any premise of a !-cell of ρ does not cover all the copy of the content of one of the boxes associated with the *box*-cells l and o of R and S. In Fig. 6, (any representative of) τ_1 (Fig. 6c) is a 1-diffnet of S (hence τ_1 is not strongly S-fat) and the ?-accessible ports from the premise of the !-cell of τ_1 cover *more* than the content of the box of *box*-cell of S: only in σ_2 (Fig. 6e), taking two copies of the content of the box, the ?-accessible ports correspond exactly to the content of the box.

A consequence of Lemma 23 (and Remark 21) is Cor. 24 below: given two box-connected MELL-ps R and S, and $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$ strongly fat, any isomorphism φ between ρ and σ "preserves" the boxes (Cor. 24.2) and the copies of the content of a box (Cor. 24.1).

▶ **Corollary 24** (Boxes and copies preservation). *Let $R, S \in \mathbf{PS}_{\mathsf{MELL}}$, $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$ with $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon \rho \simeq \sigma$. If R and S are box-connected and ρ and σ are strongly fat, then for any $(p, a), (p', a') \in \mathcal{P}_\rho$ and $(q, b), (q', b') \in \mathcal{P}_\sigma$ with $\varphi_{\mathcal{P}}((p, a)) = (q, b)$ and $\varphi_{\mathcal{P}}((p', a')) = (q', b')$:* Proof at p. 23

1. (copies preserv.) $\mathsf{box}_{\mathcal{P}_R}^{\mathsf{ext}}(p) = \mathsf{box}_{\mathcal{P}_R}^{\mathsf{ext}}(p')$ *and* $a = a'$ *iff* $\mathsf{box}_{\mathcal{P}_S}^{\mathsf{ext}}(q) = \mathsf{box}_{\mathcal{P}_S}^{\mathsf{ext}}(q')$ *and* $b = b'$*;*

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[5] See [20, Def. A.6, Rmk. A.7] for the definition of ACC for MELL-ps, which can easily be adapted to DiLL-ps: ?-cells (resp. !-cells which are not *box*-cells) are considered as generalized ⅋-cells (resp. ⊗-cells).

[6] This implies that $(l, a^-) \in \mathcal{C}_\rho^!$ and $(p, a) \in \mathsf{P}_\rho^{\mathsf{aux}}((l, a^-))$, according to the definition of $\mathsf{box}_{\mathcal{C}_R}^{\mathsf{ext}}$ in Def. 11.

2. (boxes preserv.) $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p')$ *iff* $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q')$.

Cor. 24.2 says that if two ports of ρ correspond to two ports of R in the same boxes, then their images in σ via φ corresponds to two ports of S in the same boxes, and conversely. Cor. 24.1 means that if two ports of ρ are in the same copy of the content of a box in R or correspond to ports with depth 0 in R, then their images in σ via φ are in the same copy of a box in S or correspond to ports with depth 0 in S, and conversely. The idea of the proof of Cor. 24.1 is that if two ports of ρ are in the same copy of a box in R, then (Lemma 23) they are ?-accessible from the same premise of a !-cell of ρ and thus, since ?-accessibility is preserved by isomorphism (Remark 21), their images via φ are ?-accessible from the same premise of a !-cell of σ, hence (Lemma 23 again) they are in the same copy of a box in S.

Cor. 24 (together with Fact 19) is crucial in the proof of the next lemma, which shows how to build an isomorphism ϕ between two box-connected MELL-ps R and S starting from an isomorphism φ between $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$ strongly fat: roughly speaking, ϕ is just the restriction of φ to only one copy (*e.g.* the first one) in ρ of the content of each box of R.

Proof at p. 23

▶ **Lemma 25** (Building isomorphism). *Let $R, S \in \mathbf{PS}_{\mathsf{MELL}}$, $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$. Suppose ρ and σ are strongly fat and canonical, and $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon \rho \simeq \sigma$. Let $\phi_{\mathcal{P}} \colon \mathcal{P}_R \to \mathcal{P}_S$ and $\phi_{\mathcal{C}} \colon \mathcal{C}_R \to \mathcal{C}_S$ be functions defined in Eq. (1). If R and S are box-connected, then $\phi = (\phi_{\mathcal{P}}, \phi_{\mathcal{C}}) \colon R \simeq S$.*

$$\begin{aligned} \phi_{\mathcal{P}}(p) &= \mathsf{forget}^{\sigma,S}_{\mathcal{P}}(\varphi_{\mathcal{P}}((p,a))) \quad \text{for every } p \in \mathcal{P}_R \text{ where } (p,a) \in \mathcal{P}_\rho \text{ with } a \in \{1\}^*; \\ \phi_{\mathcal{C}}(l) &= \mathsf{forget}^{\sigma,S}_{\mathcal{C}}(\varphi_{\mathcal{C}}((l,a))) \quad \text{for every } l \in \mathcal{C}_R \text{ where } (l,a) \in \mathcal{C}_\rho \text{ with } a \in \{1\}^*. \end{aligned} \tag{1}$$

▶ **Theorem 26.** *Let R and S be some box-connected MELL-ps. Let $\rho_0 \in \mathcal{T}_R^{\simeq}$ and $\sigma_0 \in \mathcal{T}_S^{\simeq}$ be strongly fat. If $\rho_0 = \sigma_0$ then $R \simeq S$.*

Proof. According to Def. 17, $\rho_0 = \sigma_0$ implies that there are $\rho \in \mathcal{T}_R \cap \rho_0$, $\sigma \in \mathcal{T}_S \cap \sigma_0$ and $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon \rho \simeq \sigma$. By Remark 14, we can suppose without loss of generality that ρ and σ are canonical. By hypothesis, ρ and σ are strongly fat. By Lemma 25, there is $\phi \colon R \simeq S$. ◀

We point out that Thm. 26 holds for any $\rho_0 \in \mathcal{T}_R^{\simeq}$ strongly fat, in particular when ρ_0 is the point of order 2 of the Taylor expansion of R, *i.e.* ρ_0 is the equivalence class of DiLL_0-ps isomorphic to a 2-diffnet of R (obtained by taking 2 copies of the content of each box in R). If R or S is not box-connected, or ρ_0 is not strongly fat, then in general $R \not\simeq S$, see Fig. 5-6.

5 Conclusion: injectivity of the relational model

Thm. 26 has a semantic counterpart: the injectivity of relational semantics for box-connected MELL-ps. The *relational model* is the simplest model of MELL; it can be seen as a degenerate case of Girard's coherent semantics [12], where formulas are interpreted as sets and proofs as relations between them. It is more or less well-known that, given a MELL-ps R, there is a correspondence between certain equivalence classes on its relational interpretation $[\![R]\!]$ and elements of its Taylor expansion $\mathcal{T}_R^{\simeq}$ (see [13], or equivalently Appendix C, for a detailed proof): in particular, if two cut-free MELL-ps with atomic axioms have the same relational semantics, then they have the same Taylor expansion. Thus, from Thm. 26 it follows that:

Proof at p. 25

▶ **Corollary 27** (Injectivity for box-connected MELL). *Let R and S be box-connected, cut-free MELL-ps with atomic axioms and conclusions of the same type. If $[\![R]\!] = [\![S]\!]$, then $R \simeq S$.*

Using different techniques, De Carvalho [4] proves the following, more general, theorem:

▶ **Theorem 28** (De Carvalho [4], injectivity for full MELL). *Let R and S be cut-free MELL-ps with atomic axioms and conclusions of the same type. If $[\![R]\!] = [\![S]\!]$, then $R \simeq S$.*

The injectivity proven in [6] is the same as Cor. 27: two different box-connected, cut-free MELL-ps with atomic axioms have different relational semantics. However, as stressed in §1, Thm. 26 (and the proof of Cor. 27) differs a lot from the proof of Thm. 28 so as from the result of [6]: [4, 6] rely on the presence, in the interpretations of MELL-ps, of points with *arbitrarily large* complexity, depending on the two MELL-ps one wishes to discriminate. On the other hand, our result allows to discriminate any two different box-connected, cut-free MELL-ps with atomic axioms using a point of the relational semantics with *fixed* complexity.

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Technical appendix

A Notations and omitted proofs and remarks

A.1 Preliminaries and notations

Formulæ We set $\mathcal{L}_{\mathsf{MELL}} = \{1, \perp, \otimes, \parr, !, ?, ax, cut\}$.

Let $\mathcal{V}_{\mathsf{MELL}}$ be a countably infinite set whose elements, denoted by $X, Y, Z, \dots$, are called *propositional variables*. The set $\mathcal{F}_{\mathsf{MELL}}$ of MELL *formulas* is generated by the grammar:

$$A, B, C ::= X \mid X^{\perp} \mid 1 \mid \perp \mid A \otimes B \mid A \parr B \mid !A \mid ?A\,.$$

If $\Gamma = (A_1, \dots, A_n)$ is a finite sequence of MELL formulas (with $n \in \mathbb{N}$), then $\parr\Gamma = A_1 \parr \cdots \parr A_n$; in particular, if $n = 0$ then $\parr\Gamma = \perp$.

For any $A \in \mathcal{F}_{\mathsf{MELL}}$, the *dual of* A, denoted by $(A)^{\perp}$ or $A^{\perp}$, is inductively defined by: $(X)^{\perp} = X^{\perp}$, $(X^{\perp})^{\perp} = X$, $(1)^{\perp} = \perp$, $(\perp)^{\perp} = 1$, $(A \otimes B)^{\perp} = (A)^{\perp} \parr (B)^{\perp}$, $(A \parr B)^{\perp} = (A)^{\perp} \otimes (B)^{\perp}$, $(!A)^{\perp} = ?(A)^{\perp}$ and $(?A)^{\perp} = !(A)^{\perp}$. Therefore, $A^{\perp\perp} = A$ for any $A \in \mathcal{F}_{\mathsf{MELL}}$.

Sequences Let $\mathcal{A}$ be a set: $\mathcal{A}^*$ is the set of finite sequences over $\mathcal{A}$. Elements of $\mathcal{A}^*$ are denoted by $(a_1, \dots, a_n)$, where $n \in \mathbb{N}$ and $a_i \in \mathcal{A}$ for any $1 \leq i \leq n$. The empty sequence is, in particular, denoted by $(\,)$; often $(a_1) \in \mathcal{A}^*$ is denoted by a_1.

If $a = (a_1, \dots, a_n)$ with $n \in \mathbb{N}$, we set $|a| = n$ and, if $n > 0$, $a^- = (a_1, \dots, a_{n-1})$.

Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_m)$ with $n, m \in \mathbb{N}$: the *concatenation* of a and b is $a \cdot b = (a_1, \dots, a_n, b_1, \dots, b_m)$. We write $a \sqsubseteq b$ if $a \cdot c = b$ for some finite sequence c.

Sets and orders Let $\mathcal{A}$ be a set: $\mathsf{card}(\mathcal{A})$ is the cardinality of $\mathcal{A}$, $\mathscr{P}(\mathcal{A})$ is the power set of $\mathcal{A}$, $\mathscr{P}_{\mathrm{fin}}(\mathcal{A})$ is the set of finite subsets of $\mathcal{A}$, $\bigcup \mathcal{A}$ is the union of the elements of $\mathcal{A}$.

A preorder on $\mathcal{A}$ is a reflexive and transitive binary relation on $\mathcal{A}$. An order $\leq$ on $\mathcal{A}$ is an antisymmetric preorder; we say then that $\mathcal{A}$ is ordered by $\leq$.

Let $\mathcal{A}$ be ordered by $\leq$. For every $a \in \mathcal{A}$, we set $\downarrow_{\leq} a = \{b \in \mathcal{A} \mid b \leq a\}$; if no ambiguity arises, $\downarrow_{\leq} a$ is also denoted by $\downarrow_{\mathcal{A}} a$. We say that $\leq$ is a tree-like order if, for any $a \in \mathcal{A}$, $\downarrow_{\mathcal{A}} a$ is a finite subset of $\mathcal{A}$ totally ordered by $\leq$.

Functions Let $\mathcal{A}, \mathcal{B}$ be sets and $\mathsf{f} \colon \mathcal{A} \to \mathcal{B}$ be a function (resp. partial function): we set $\mathsf{dom}(\mathsf{f}) = \mathcal{A}$ (resp. $\mathsf{dom}(\mathsf{f}) = \{a \in \mathcal{A} \mid \mathsf{f}(a) \text{ is defined}\}$) the *domain* of f, and $\mathsf{im}(\mathsf{f}) = \{\mathsf{f}(a) \mid a \in \mathsf{dom}(\mathsf{f})\}$ the *image* of f. Given $\mathcal{A}' \subseteq \mathcal{A}$, the function (resp. partial function) $\mathsf{f}{\restriction_{\mathcal{A}'}} \colon \mathcal{A}' \to \mathcal{B}$ is defined by $\mathsf{dom}(\mathsf{f}{\restriction_{\mathcal{A}'}}) = \mathsf{dom}(\mathsf{f}) \cap \mathcal{A}'$ and $\mathsf{f}{\restriction_{\mathcal{A}'}}(a) = \mathsf{f}(a)$ for any $a \in \mathsf{dom}(\mathsf{f}{\restriction_{\mathcal{A}'}})$.

The function (resp. partial function) $\bar{\mathsf{f}} \colon \mathscr{P}(\mathcal{A}) \to \mathscr{P}(\mathcal{B})$ (the covariant powerset lifting of f) is defined by $\bar{\mathsf{f}}(\mathcal{A}') = \{\mathsf{f}(a) \mid a \in \mathcal{A}' \cap \mathsf{dom}(\mathsf{f})\}$ for any $\mathcal{A}' \subseteq \mathcal{A}$.

A.2 Omitted proofs and remarks of Section 2

Stated at p. 5
▶ **Fact 4** (Tree-like order on ports). *Let Φ be a pps: $\leq_{\Phi}$ is a tree-like order relation on $\mathcal{P}_{\Phi}$.*

Proof. Let $p, q \in \mathcal{P}_{\Phi}$. By Remark 3, if $p \leq_{\Phi} q$ and $q \leq_{\Phi} p$ then $p = q$. So, $\leq_{\Phi}$ is antisymmetric and thus an order, according to Definition 2.

Let $p, q, r \in \mathcal{P}_{\Phi}$ with $p \leq_{\Phi} r$ and $q \leq_{\Phi} r$. If $p \not\leq_{\Phi} q$ and $q \not\leq_{\Phi} p$ then there would be $p', q', r' \in \mathcal{P}_{\Phi}$ such that $p' \neq q'$, $p' <^1_{\Phi} r'$ and $q' <^1_{\Phi} r'$, but this is impossible by Remark 3. Thus, either $p \leq_{\Phi} q$ or $q \leq_{\Phi} p$. So, for any $r \in \mathcal{P}_{\Phi}$, the set $\downarrow_{\mathcal{P}_{\Phi}} r$ is totally ordered (and finite since $\mathcal{P}_{\Phi}$ is finite). ◀

▶ **Remark 29.** Let Φ be a pps. It follows directly from Def. 1 and 8 that:

1. $\mathcal{D}oors_\Phi = \mathcal{D}oors^!_\Phi \cup \mathcal{D}oors^?_\Phi \cup \mathcal{D}oors^{cut}_\Phi$, where $\mathcal{D}oors^!_\Phi$, $\mathcal{D}oors^?_\Phi$, $\mathcal{D}oors^{cut}_\Phi$ are pairwise disjoint; moreover, $\mathcal{D}oors_\Phi \neq \emptyset$ iff $\mathcal{C}^{\mathsf{box}}_\Phi \neq \emptyset$ iff there is $p \in \mathcal{P}_\Phi$ such that $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(p) \neq \bullet$;
2. $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(p) = \mathsf{box}_\Phi(p)$ for any $p \in \mathcal{D}oors_\Phi$;
3. for all $l \in \mathcal{C}_\Phi$ and $p, q \in \mathcal{P}_\Phi$ with $p \in \mathsf{P}^{\mathsf{pri}}_\Phi(l)$ and $q \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$, if $l \notin \mathcal{C}^{\mathsf{bord}}_\Phi$ then $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(q)$; the converse does not hold in general (see the pps Ψ_1 in Figure 1b: the *box*-cell $l \in \mathcal{C}^{\mathsf{bord}}_\Psi$ but $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_{\Psi_1}}(p) = l = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_{\Psi_1}}(q)$);
4. for every $p \in \mathcal{P}_\Phi$, if $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(p) = \bullet$ then $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(q) = \bullet$ for all $q \in \downarrow_{\mathcal{P}_\Phi} p$;
5. $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_\Phi}(p) = \bullet$ for all $p \in \mathcal{P}^{\mathsf{free}}_\Phi$, since the conclusions of Φ are minimal elements of $\mathcal{P}_\Phi$ with respect to $\leq_\Phi$ and $\mathcal{P}^{\mathsf{free}}_\Phi \cap \mathcal{D}oors_\Phi = \emptyset$;
6. for every $p \in \bigcup \overline{\mathsf{P}^{\mathsf{aux}}_\Phi}(\mathcal{C}^{cut}_\Phi)$, $\mathsf{box}^{\mathsf{ext}}_\Phi(p) = \bullet$ if and only if $p \notin \mathcal{D}oors^{cut}_\Phi$.

▶ **Remark 30.** Let R a DiLL-ps.

1. By condition 3 in Def. 10, the converse of Remark 29.3 holds in R: for all $l \in \mathcal{C}_R$ with $p \in \mathsf{P}^{\mathsf{pri}}_R(l)$ and $q \in \mathsf{P}^{\mathsf{aux}}_R(l)$, if $l \in \mathcal{C}^{\mathsf{bord}}_R$ then $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) \neq \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(q)$.
2. By conditions 1 and 3-4 in Def. 10, a *box*-cell is in the border of exactly one box: for any $l \in \mathcal{C}^{\mathsf{box}}_R$ with $\mathsf{P}^{\mathsf{pri}}_R(l) = \{p\}$, one has $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) <_{\mathcal{C}^{\mathsf{box}}_R \cup \{\bullet\}} l$ and there is no $l' \in \mathcal{C}^{\mathsf{box}}_R \cup \{\bullet\}$ such that $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) <_{\mathcal{C}^{\mathsf{box}}_R \cup \{\bullet\}} l' <_{\mathcal{C}^{\mathsf{box}}_R \cup \{\bullet\}} l$. This does not hold in general for ?-cells in $\mathcal{C}^{\mathsf{bord}}_R$, since we use generalized ?-links: a premise of a ?-cell can cross the border of several boxes, see for instance one of the premises of the ?-cell whose conclusion is of type $?\bot$ in Fig. 3a.

A.3 Omitted proofs and remarks of Section 3

In the original definition by [10] the Taylor expansion of a λ-term (of which the Taylor expansion of DiLL-ps is a generalization) is a (usually infinite) linear combination of resource λ-terms with scalars in some semiring. With respect to this paper scalars play no role, thus we have defined the Taylor expansion of a DiLL-ps as just a set of DiLL_0-ps, as in [16, 18].

▶ **Remark 31.** Let R be a DiLL-ps and $\rho \in \mathcal{T}_R$.

1. According to Remarks 29.3 and 30.1, for any $(l, a) \in \mathcal{C}_\rho$ one has $\mathsf{P}^{\mathsf{aux}}_\rho((l, a)) = \{(p, a) \mid p \in \mathsf{P}^{\mathsf{aux}}_R(l)\}$ iff $l \notin \mathcal{C}^{\mathsf{bord}}_R$. This ensures, in particular, that $\mathsf{P}^{\mathsf{left}}_{\tau_R(\mathsf{f})}$ is well-defined in Def. 13.
2. By Remark 30.2 and the definition of $\mathsf{box}^{\mathsf{ext}}_{\mathcal{C}_R}$ in Def. 11, if $l \in \mathcal{C}^{\mathsf{box}}_R$ and $(\mathsf{prid}_R(l), a) \in \mathcal{P}_\rho$, then $(l, a^-) \in \mathcal{C}^!_\rho$ and $(\mathsf{prid}_R(l), a)$ is a premise of (l, a^-); if moreover $q \in \mathsf{inbox}_R(l)$, $b \in \mathbb{N}^*$ and $(q, b) \in \mathcal{P}_\rho$, then $a \sqsubseteq b$ thanks to the vertical downclosure condition of Def. 12.

▶ **Remark 32.** Let R be a DiLL-ps and f be a Taylor-function of R, with $\rho = \tau_R(\mathsf{f}) \in \mathcal{T}_R$. These facts follow immediately from Def. 12-13:

1. Any port or cell of ρ comes from a port or cell of R, *i.e.* if $q \in \mathcal{P}_\rho$ then $q = (p, a)$ where $p \in \mathcal{P}_R$, $a \in \mathsf{f}(\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p))$ and $|a| = \mathsf{depth}_R(p)$; in particular, if $(p, ()) \in \mathcal{P}_\rho$ then $p \in \mathcal{P}^0_R$. Analogously for the cells of ρ. Moreover, the corresponding ports and cells of ρ and R "play the same role": for instance, given a cell (l, a) of ρ having (p, a) as a conclusion, then l is a cell of R having p as a conclusion.
2. Ports of R contained in the same boxes have in ρ same indexes, *i.e.* for all $a \in \mathbb{N}^*$ and $p, q \in \mathcal{P}_R$ such that $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(q)$, $(p, a) \in \mathcal{P}_\rho$ iff $(q, a) \in \mathcal{P}_\rho$. Analogously for cells of R. Also, the indexes of any cell of ρ and the ones in its conclusions are the same, *i.e.* if $(l, b) \in \mathcal{C}_\rho$ and $(p, a) \in \mathsf{P}^{\mathsf{pri}}_\rho((l, b))$ then $a = b$ (see definition of $\mathsf{box}^{\mathsf{ext}}_{\mathcal{C}_R}$ in Def. 11).

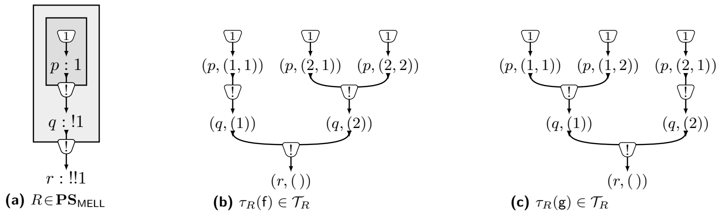


Figure 7 A MELL-ps R (Fig. 7a, where o and l are the *box*-cells with conclusion of type $!1$ and $!!1$, respectively), and two different but isomorphic elements $\tau_R(\mathsf{f})$ (Fig. 7b) and $\tau_R(\mathsf{g})$ (Fig. 7c) of $\mathcal{T}_R$, where f and g are the Taylor-functions of R defined in Example 33.

3. Due to depth compatibility condition of Def. 12, $p \in \mathcal{P}_R^0$ (resp. $l \in \mathcal{C}_R^0$) iff $(p,()) \in \mathcal{P}_\rho$ (resp. $(l,()) \in \mathcal{C}_\rho$). In particular, there is a one-to-one correspondence between free ports (resp. terminal cells) of R and free ports (resp. terminal cells) of ρ.

Given $R \in \mathbf{PS}_{\mathsf{DiLL}}$ and $\mathsf{f} \in \mathcal{T}_R^{\mathsf{proto}}$ such that $\rho = \tau_R(\mathsf{f}) \in \mathcal{T}_R$, the functions $\mathsf{f} \circ \mathsf{box}_{\mathcal{P}_R}^{\mathsf{ext}}$ and $\mathsf{f} \circ \mathsf{box}_{\mathcal{C}_R}^{\mathsf{ext}}$ are some kind of "inverses" of $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ and $\mathsf{forget}_{\mathcal{C}}^{\rho,R}$, respectively: with every port and cell of R, they associate the set of indexes of their corresponding ports and cells of ρ. In other words, for every port p and cell l of R, $\mathsf{f}(\mathsf{box}_{\mathcal{P}_R}^{\mathsf{ext}}(p))$ and $\mathsf{f}(\mathsf{box}_{\mathcal{C}_R}^{\mathsf{ext}}(l))$ are the sets of the "tracking numbers" of the copies of (the content of the boxes containing) p and l in ρ.

▶ **Example 33.** Let R be the MELL-ps as in Figure 7a and let f and g be the Taylor-functions of R such that $\mathsf{f}(l) = \{(1),(2)\}$, $\mathsf{f}(o) = \{(1,1),(2,1),(2,2)\}$, $\mathsf{g}(l) = \{(1),(2)\}$ and $\mathsf{g}(o) = \{(1,1),(1,2),(2,1)\}$. Obviously, $\mathsf{f} \neq \mathsf{g}$, $\tau_R(\mathsf{f}) \neq \tau_R(\mathsf{g})$ (indeed, $(p,(2,2)) \in \mathcal{P}_{\tau_R(\mathsf{f})} \smallsetminus \mathcal{P}_{\tau_R(\mathsf{g})}$, see Fig. 7b-7c) but $\tau_R(\mathsf{f}) \simeq \tau_R(\mathsf{g})$ (and $\tau_R(\mathsf{f}), \tau_R(\mathsf{g}) \in \mathcal{T}_R$).

▶ **Definition 34** (Pruning of a Taylor-function)**.** Let R be a DiLL-ps. Given $\mathsf{f},\mathsf{g} \in \mathcal{T}_R^{\mathsf{proto}}$, g is a *pruning* of f if $\mathsf{g}(l) \subseteq \mathsf{f}(l)$ for all $l \in \mathcal{C}_R^{\mathsf{box}}$.

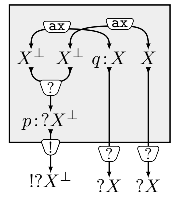
In the tree-like representation of Taylor-functions (see Fig. 4c), a pruning of a Taylor-function f of R is obtained from f by removing some branches.

▶ **Remark 35.** Let R be a DiLL-ps and $\mathsf{f} \in \mathcal{T}_R^{\mathsf{proto}}$, with $\rho = \tau_R(\mathsf{f}) \in \mathcal{T}_R$.

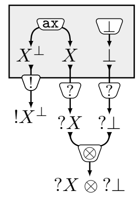
1. If R is a MELL-ps, one has: ρ is fat (resp. strongly fat) iff ρ is R-fat (resp. strongly R-fat), since in a MELL-ps !-cells and *box*-cells coincide.
2. If ρ is R-fat and $p \in \mathcal{P}_R$ (resp. $l \in \mathcal{C}_R$), then there is some $a \in \mathbb{N}^*$ such that $(p,a) \in \mathcal{P}_\rho$ (resp. $(l,a) \in \mathcal{C}_\rho$); if moreover ρ is canonical then such a a can be chosen such that $a \in \{1\}^*$. This is false in general if ρ is not R-fat: given $l \in \mathcal{C}_R^{\mathsf{box}}$, if for all !-cells in ρ corresponding to l one takes 0 copies of the content of the box of l, then $(\mathsf{prid}_R(l), a) \notin \mathcal{P}_\rho$ for all $a \in \mathbb{N}^*$.
3. By the vertical downclosure condition in Def. 12, $\tau_R(\mathsf{f})$ is R-fat (resp. strongly R-fat) iff for any $l \in \mathcal{C}_R^{\mathsf{box}}$ and $a \in \mathsf{f}(\mathsf{box}_{\mathcal{C}_R}^{\mathsf{ext}}(l))$, $a \cdot n \in \mathsf{f}(l)$ (resp. $a \cdot n, a \cdot m \in \mathsf{f}(l)$) for some $n \in \mathbb{N}$ (resp. $n, m \in \mathbb{N}$ with $n \neq m$). Given $k \in \mathbb{N}$, $\tau_R(\mathsf{f})$ is a k-diffnet of R iff for any $l \in \mathcal{C}_R^{\mathsf{box}}$ and $a \in \mathsf{f}(\mathsf{box}_{\mathcal{C}_R}^{\mathsf{ext}}(l))$ there are pairwise distinct $n_1, \dots, n_k \in \mathbb{N}$ such that $\mathsf{f}(l) = \{a \cdot n_1, \dots, a \cdot n_k\}$.
4. If $\tau_R(\mathsf{f})$ is R-fat, then there is a pruning $\mathsf{g} \in \mathcal{T}_R^{\mathsf{proto}}$ of f such that $\tau_R(\mathsf{g})$ is a 1-diffnet of R.

Stated at p. 11 ▶ **Fact 19** (Isomorphisms between ground-structures)**.** *Let R, S be DiLL-ps, and ρ (resp. σ) be a 1-diffnet of R (resp. S).*

1. $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ *and* $\mathsf{forget}_{\mathcal{C}}^{\rho,R}$ *are bijections, and* $(\mathsf{forget}_{\mathcal{P}}^{\rho,R}, \mathsf{forget}_{\mathcal{C}}^{\rho,R})\colon G_\rho \simeq G_R$.
2. *Suppose $\varphi_1 \colon \rho \simeq \sigma$. Let $\varphi_{\mathcal{P}} \colon \mathcal{P}_R \to \mathcal{P}_S$ and $\varphi_{\mathcal{C}} \colon \mathcal{C}_R \to \mathcal{C}_S$ be functions defined by (for all $p \in \mathcal{P}_R$, $l \in \mathcal{C}_R$ and $a, b \in \mathbb{N}^*$ with $(p,a) \in \mathcal{P}_\rho$ and $(l,b) \in \mathcal{C}_\rho$): $\varphi_{\mathcal{P}}(p) = \mathsf{forget}_{\mathcal{P}}^{\sigma,S}(\varphi_{1\mathcal{P}}((p,a)))$ and $\varphi_{\mathcal{C}}(l) = \mathsf{forget}_{\mathcal{C}}^{\sigma,S}(\varphi_{1\mathcal{C}}((l,b)))$. Then, $\varphi_{\mathcal{P}}$ and $\varphi_{\mathcal{C}}$ are bijective and $(\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}})\colon G_R \simeq G_S$.*



(a) A box-connected MELL-ps R



(b) A non-box-connected MELL-ps S

Figure 8 A box-connected DiLL-ps R (Fig. 8a) and a non-box-connected DiLL-ps S (Fig. 8b). The port p of R is not ?-accessible from the port q of R, but q is ?-accessible from p.

Proof.

1. Let $\mathsf{f} \in \mathcal{T}_R^{\mathsf{proto}}$ be such that $\rho = \tau_R(\mathsf{f})$. By Remark 35.3, $\mathsf{card}(\mathsf{f}(l))) = 1$ for any $l \in \mathcal{C}_R^{\mathsf{box}}$, thus $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ and $\mathsf{forget}_{\mathcal{C}}^{\rho,R}$ are bijective. By construction (see also Remark 32.1), $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ and $\mathsf{forget}_{\mathcal{C}}^{\rho,R}$ make diagrams in Fig. 2a (see Def. 5) commute, hence $(\mathsf{forget}_{\mathcal{P}}^{\rho,R}, \mathsf{forget}_{\mathcal{C}}^{\rho,R})\colon G_\rho \simeq G_R$.
2. By Remark 35.2, the functions $\varphi_{\mathcal{P}}$ and $\varphi_{\mathcal{C}}$ are defined for any $p \in \mathcal{P}_R$ and $l \in \mathcal{C}_R$, respectively, since ρ is R-fat. From Fact 19.1 and since $\varphi_1\colon \rho \simeq \sigma$, it follows that $\varphi_{\mathcal{P}}$ and $\varphi_{\mathcal{C}}$ are bijections and make diagram in Fig. 2b (see Def. 5) commute, by composition. Therefore $(\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}})\colon G_R \simeq G_S$. ◀

▶ **Definition 36** (Order of an element of the Taylor expansion)**.** Let R be a DiLL-ps and $k \in \mathbb{N}$. A $\rho_0 \in \mathcal{T}_R^{\simeq}$ is the *element of order k of $\mathcal{T}_R^{\simeq}$* if $\rho \in \rho_0$ for some k-diffnet ρ of R.

Recall that the elements of the Taylor expansion of a DiLL-ps are equivalences classes of DiLL_0-ps modulo isomorphism (Def. 17). According to Remark 6.1, given a DiLL-ps R, if ρ_0 is the element of order k of $\mathcal{T}_R^{\simeq}$, then all $\rho \in \rho_0$ are isomorphic to any k-diffnet of R. Clearly, for any $k \in \mathbb{N}$ the element of order k of $\mathcal{T}_R^{\simeq}$ exists and is unique. Roughly speaking, (any representative of) the element of order k of $\mathcal{T}_R^{\simeq}$ is obtained by taking exactly k copies of the content of each box in R.

A.4 Omitted proofs and remarks of Section 4

Note that $\mathcal{L}$-accessibility cannot be defined as a binary symmetric relation on the ports of a pps Φ: in general, $q \in \mathsf{acces}_\Phi^?(p)$ does not imply that $p \in \mathsf{acces}_\Phi^?(q)$, as exemplified by the MELL-ps R in Fig. 8a with $\mathcal{L} = \{?\}$.

▶ **Remark 37.** For any DiLL-ps R and $l \in \mathcal{C}_R^{\mathsf{box}}$, if R is box-connected then $\mathsf{inbox}_R(l) \subseteq \mathsf{acces}_R^?(\mathsf{prid}_R(l))$. The converse is false: in the non-box-connected MELL-ps S in Fig. 8b, the only *box*-cell l is such that $\mathsf{inbox}_R(l) \subseteq \mathsf{acces}_R^?(\mathsf{prid}_R(l))$.

The next lemma shows how, given a box B in a DiLL-ps R, the $\mathcal{L}$-accessible ports in B from the *pri*-door p of B are related to the $\mathcal{L}$-accessible ports from a port corresponding to p in a R-fat $\rho \in \mathcal{T}_R$.

▶ **Lemma 38.** *Let $R \in \mathbf{PS}_{\mathsf{DiLL}}$, let $\rho \in \mathcal{T}_R$ be R-fat and $(p, a) \in \mathcal{P}_\rho$. For any $l \in \mathcal{C}_R^{\mathsf{box}}$, if $\mathsf{prid}_R(l) = p$, then $\mathsf{acces}_\rho^?((p,a)) \supseteq \{(q, a{\cdot}b) \in \mathcal{P}_\rho \mid b \in \mathbb{N}^*,\ q \in \mathsf{inbox}_R(l)$ and there is a ?-path in R from p to q inside the box of $l\}$.*

Proof. Let $(q, a\cdot b) \in \mathcal{P}_\rho$ for some $b \in \mathbb{N}^*$ and $q \in \mathcal{P}_R$ such that there is a ?-path in R from p to q inside the box of l. We prove that there is a ?-path in ρ from (p, a) to $(q, a\cdot b)$ (and then $(q, a\cdot b) \in \mathsf{acces}^?_\rho((p, a)))$ by induction on the definition of the ?-path $\vec{r}$ in R from p to q. Consider the last rule in the construction of $\vec{r}$ (we follow the enumeration of Def. 20).

(i) $\vec{r} = (p)$, thus $q = p$ and (by depth compatibility condition (1) in Def. 12) $b = (\,)$, so $(q, a\cdot b) = (p, a)$ and there is a ?-path in ρ from (p, a) to $(q, a\cdot b)$ according to the rule (i) of Def. 20.

(ii) $\vec{r} = (p_0, \dots, p_n, q)$ with $p_0 = p$ and $p_n \in \mathsf{P}^{\mathsf{pri}}_R(l')$ for some $l' \in \mathcal{C}_R$ such that $p_n \neq q \in \mathsf{P}^{\mathsf{pri}}_R(l') \cup \mathsf{P}^{\mathsf{aux}}_R(l')$. By Remarks 31.2 and 35.2-3, since ρ is R-fat and $p_n, q \in \mathsf{inbox}_R(l)$, there is $c \in \mathbb{N}^*$ such that $c \sqsubseteq b$, $(p_n, a\cdot c) \in \mathsf{P}^{\mathsf{pri}}_\rho((l', a\cdot c))$ and $(q, a\cdot b) \in \mathsf{P}^{\mathsf{aux}}_\rho((l', a\cdot c))$. By induction hypothesis, there is a ?-path $\vec{s}$ in ρ from (p, a) to $(p_n, a\cdot c)$, and hence, according to the rule (ii) of Def. 20, $\vec{s}\cdot(q, a\cdot b)$ is a ?-path in ρ from (p, a) to $(q, a\cdot b)$.

(iii) $\vec{r} = (p_0, \dots, p_n, q)$ with $p_0 = p$ and $p_n \in \mathsf{P}^{\mathsf{aux}}_R(l')$ for some $l' \in \mathcal{C}_R$ such that $\mathsf{tc}(l') \neq ?$ and $p_n \neq q \in \mathsf{P}^{\mathsf{pri}}_R(l') \cup \mathsf{P}^{\mathsf{aux}}_R(l')$. By Remarks 31.2 and 35.2-3, since ρ is R-fat and $p_n, q \in \mathsf{inbox}_R(l)$, one has $(l', a\cdot b) \in \mathcal{C}_\rho$, $(q, a\cdot b) \in \mathsf{P}^{\mathsf{pri}}_\rho((l', a\cdot b))$ and there is $c \in \mathbb{N}^*$ such that $b \sqsubseteq c$ and $(p_n, a\cdot c) \in \mathsf{P}^{\mathsf{aux}}_\rho((l', a\cdot b))$. By induction hypothesis, there is a ?-path $\vec{s}$ in ρ from (p, a) to $(p_n, a\cdot c)$, and hence, by applying the rule (iii) of Def. 20, $\vec{s}\cdot(q, a\cdot b)$ is a ?-path in ρ from (p, a) to $(q, a\cdot b)$.

(iv) $\vec{r} = (p_0, \dots, p_n, q)$ with $p_0 = p$ and $p_n \in \mathsf{P}^{\mathsf{aux}}_R(l')$ for some $l' \in \mathcal{C}^?_R$ such that $p_n \neq q \in \mathsf{P}^{\mathsf{pri}}_R(l') \cup \mathsf{P}^{\mathsf{aux}}_R(l')$ and, for any $p' \in \mathsf{P}^{\mathsf{aux}}_R(l')$, there is a ?-path from p_0 to p'. By Remarks 31.2 and 35.2-3, since ρ is R-fat and $p_n, q \in \mathsf{inbox}_R(l)$, one has $(l', a\cdot b) \in \mathcal{C}_\rho$ and, for any $p' \in \mathsf{P}^{\mathsf{aux}}_R(l')$ (in particular for $p' = p_n$), there is $c \in \mathbb{N}^*$ such that $b \sqsubseteq c$, $(p', a\cdot c) \in \mathsf{P}^{\mathsf{aux}}_\rho((l', a\cdot b))$ and $(q, a\cdot b) \in \mathsf{P}^{\mathsf{pri}}_\rho((l', a\cdot b))$. By induction hypothesis, there is a ?-path in ρ from (p, a) to $(p', a c)$ for any $p' \in \mathsf{P}^{\mathsf{aux}}_R(l')$. If there where some $(q', d) \in \mathsf{P}^{\mathsf{aux}}_\rho(l')$ such that $a \not\sqsubseteq d$ then $l' \in \mathcal{C}^{\mathsf{bord}}_R$ and $q \notin \mathsf{inbox}_R(l)$, that is impossible. Hence, there is a ?-path in ρ from (p, a) to $(q, a\cdot b)$ by applying the rule (iv) of Def. 20. ◀

▶ **Remark 39.** Let R be a DiLL-ps and $\mathsf{f} \in \mathcal{T}^{\mathsf{proto}}_R$, with $\rho = \tau_R(\mathsf{f}) \in \mathcal{T}_R$. From Remarks 35.2-3, it follows that, given $l \in \mathcal{C}^{\mathsf{box}}_R$ and $p \in \mathsf{inbox}_R(l)$, if ρ is R-fat and $\mathsf{card}(\mathsf{f}(l)) \geq 2$, then there are $b, c \in \mathbb{N}^*$ such that $b \neq c$, $b \not\sqsubseteq c$, $c \not\sqsubseteq b$ and $(p, b), (p, c) \in \mathcal{P}_\rho$; and if moreover $\mathsf{pred}_R(p) \in \mathcal{P}_R \smallsetminus \mathsf{inbox}_R(l)$ then, for any $a \in \mathbb{N}^*$ such that $(\mathsf{prid}_R(l), a) \in \mathcal{P}_\rho$, either $a \not\sqsubseteq b$ or $a \not\sqsubseteq c$.

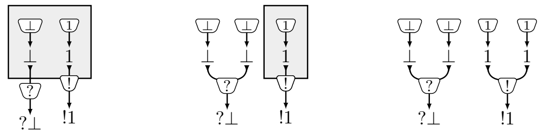
▶ **Lemma 23** (Geometric characterization of the copies of the content of boxes in an element of the labelled Taylor expansion). *Let R be a DiLL-ps, let $\rho \in \mathcal{T}_R$ and $(p, a) \in \mathcal{P}_\rho$ with $p = \mathsf{prid}_R(l)$ for some $l \in \mathcal{C}^{\mathsf{box}}_R$. Let $\mathcal{P}^{l,a}_\rho = \{(q, a\cdot b) \in \mathcal{P}_\rho \mid b \in \mathbb{N}^* \text{ and } q \in \mathsf{inbox}_R(l)\}$. (Recall that $(l, a^-) \in \mathcal{C}^!_\rho$ and $(p, a) \in \mathsf{P}^{\mathsf{aux}}_\rho((l, a^-))$, according to Remark 31.2 above).* Stated at p. 13

1. *If $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\rho((l, a^-))) \geq 2$ (in particular, if ρ is strongly R-fat), then $\mathsf{acces}^?_\rho((p, a)) \subseteq \mathcal{P}^{l,a}_\rho$.*
2. *If ρ is R-fat and R is box-connected, then $\mathcal{P}^{l,a}_\rho \subseteq \mathsf{acces}^?_\rho((p, a))$.*
3. *If R is box-connected, and if ρ is R-fat and $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\rho((l, a^-))) \geq 2$ (in particular, if ρ is strongly R-fat), then $\mathcal{P}^{l,a}_\rho = \mathsf{acces}^?_\rho((p, a))$ and thus $\mathsf{inbox}_R(l) = \overline{\mathsf{forget}^{\rho,R}_{\mathcal{P}}}(\mathsf{acces}^?_\rho((p, a)))$.*

Proof.

1. We prove a stronger statement: if $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\rho((l, a^-))) \geq 2$ then any ?-path $\vec{r} = (r_0, \dots, r_n)$ in ρ with $r_0 = (p, a)$ is such that $r_i \in \mathcal{P}^{l,a}_\rho$ for any $0 \leq i \leq n$. The proof is by induction on the definition $\vec{r}$. Consider the last rule in the construction of $\vec{r}$ (we follow the enumeration of Def. 20).

   (i) $\vec{r} = ((p, a))$, then $n = 0$ and $r_0 = (p, a) \in \mathcal{P}^{l,a}_\rho$.



(a) $R \in \mathbf{PS}_{\mathsf{MELL}}$ **(b)** $S \in \mathbf{PS}_{\mathsf{MELL}}$ **(c)** $\tau \in \mathcal{T}_R^{\simeq} \cap \mathcal{T}_S^{\simeq}$

Figure 9 Two non-isomorphic MELL-ps R (Fig. 9a) and S (Fig. 9b), having the same element τ of order 2 in their Taylor expansions (Fig. 9c, see Def. 36). Note that R is not box-connected.

(ii) $\vec{r} = (r_0, \dots, r_n, r_{n+1})$ with $r_0 = (p, a)$ and $r_n \in \mathsf{P}^{\mathsf{pri}}_\rho(l')$ for some $l' \in \mathcal{C}_\rho$ such that $r_n \neq r_{n+1} \in \mathsf{P}^{\mathsf{pri}}_\rho(l') \cup \mathsf{P}^{\mathsf{aux}}_\rho(l')$. By induction hypothesis, $r_i \in \mathcal{P}^{l,a}_\rho$ for any $1 \leq i \leq n$, in particular $r_n = (q', a \cdot b')$ for some $b' \in \mathbb{N}^*$ and $q' \in \mathsf{inbox}_R(l)$. By construction, $r_{n+1} = (q, a \cdot b)$ for some $q \in \mathsf{inbox}_R(l)$ and $b \in \mathbb{N}^*$, thus $r_{n+1} \in \mathcal{P}^{l,a}_\rho$.

(iii) $\vec{r} = (r_0, \dots, r_n, r_{n+1})$ with $(p, a) = r_0 \neq r_n \in \mathsf{P}^{\mathsf{aux}}_\rho(l')$ for some $l' \in \mathcal{C}_\rho \smallsetminus \mathcal{C}^?_\rho$ such that $r_n \neq r_{n+1} \in \mathsf{P}^{\mathsf{pri}}_\rho(l') \cup \mathsf{P}^{\mathsf{aux}}_\rho(l')$. By induction hypothesis, $r_i \in \mathcal{P}^{l,a}_\rho$ for any $1 \leq i \leq n$, in particular $r_n = (q', a \cdot b')$ for some $b' \in \mathbb{N}^*$ and $q' \in \mathsf{inbox}_R(l)$. By construction, $l' = (l'', c)$ for some $c \in \mathbb{N}^*$ and $l'' \in \mathcal{C}_R$: since $l' \notin \mathcal{C}^?_\rho$, then $l'' \notin \mathcal{C}^?_R$. Moreover $l'' \neq l$, otherwise (as $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_R(l)) = 1$ and $r_0 \neq r_n$) $r_n = (p, d)$ for some $d \in \mathbb{N}^*$ such that $a \not\sqsubseteq d$, that is impossible since $r_n = (q', a \cdot b')$. Hence, l'' is not a cell on the border of the box of l, and so $r_{n+1} = (q, a \cdot b)$ for some $q \in \mathsf{inbox}_R(l)$ and $b \in \mathbb{N}^*$, thus $r_{n+1} \in \mathcal{P}^{l,a}_\rho$.

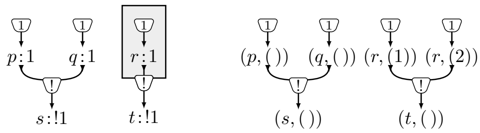
(iv) $\vec{r} = (r_0, \dots, r_n, r_{n+1})$ with $(p, a) = r_0 \neq r_n \in \mathsf{P}^{\mathsf{aux}}_\rho(l')$ for some $l' \in \mathcal{C}^?_\rho$ such that $r_n \neq r_{n+1} \in \mathsf{P}^{\mathsf{pri}}_\rho(l') \cup \mathsf{P}^{\mathsf{aux}}_\rho(l')$ and, for any $r' \in \mathsf{P}^{\mathsf{aux}}_\rho(l')$, there is a ?-path from r_0 to r'. By induction hypothesis, $\mathsf{P}^{\mathsf{aux}}_\rho(l') \subseteq \mathcal{P}^{l,a}_\rho$, so for any $r' \in \mathsf{P}^{\mathsf{aux}}_\rho(l)$ (in particular, for $r' = r_n$) there are $b' \in \mathbb{N}^*$ and $q' \in \mathsf{inbox}_R(l)$ such that $r' = (q', a \cdot b')$. By construction, $l' = (l'', c)$ and $r_{n+1} = (q, c)$ for some $c \in \mathbb{N}^*$, $l'' \in \mathcal{C}^?_R$ and $q \in \mathsf{P}^{\mathsf{pri}}_\rho(l') \cup \mathsf{P}^{\mathsf{aux}}_R(l'')$. By Remark 39, $q \in \mathsf{inbox}_R(l)$ and $a \sqsubseteq c$, *i.e.* $c = a \cdot b$ for some $b \in \mathbb{N}^*$. Hence, l'' is not a cell on the border of the box of l, and so $r_{n+1} = (q, a \cdot b) \in \mathcal{P}^{l,a}_\rho$.

2. By Lemma 38, $\mathsf{acces}^?_\rho((p, a)) \supseteq \{(q, a \cdot b) \in \mathcal{P}_\rho \mid b \in \mathbb{N}^*,\ q \in \mathsf{inbox}_R(l)$ and there is a ?-path in R from p to q inside the box of $l\} = \mathcal{P}^{l,a}_\rho$, according to the definition of box-connected (Def. 22).

3. From Lemmas 23.1-2 it follows that $\mathcal{P}^{l,a}_\rho = \mathsf{acces}^?_\rho((p, a))$. Thus, $\overline{\mathsf{forget}^{\rho,R}_{\mathcal{P}}}(\mathsf{acces}^?_\rho((p, a))) \subseteq \mathsf{inbox}_R(l)$ by definition of $\mathcal{P}^{l,a}_\rho$ and $\mathsf{forget}^{\rho,R}_{\mathcal{P}}$ (Def. 16). Conversely, given $q \in \mathsf{inbox}_R(l)$, then there exists $b \in \mathbb{N}^*$ such that $(q, b) \in \mathcal{P}_\rho$ by Remark 35.2, since ρ is R-fat; according to Remark 31.2, $a \sqsubseteq b$ and hence $(q, b) \in \mathcal{P}^{l,a}_\rho = \mathsf{acces}^?_\rho((p, a))$, thus $q = \mathsf{forget}^{\rho,R}_{\mathcal{P}}((q, b)) \in \overline{\mathsf{forget}^{\rho,R}_{\mathcal{P}}}(\mathsf{acces}^?_\rho((p, a)))$. Therefore, $\mathsf{inbox}_R(l) \subseteq \overline{\mathsf{forget}^{\rho,R}_{\mathcal{P}}}(\mathsf{acces}^?_\rho((p, a)))$. ◀

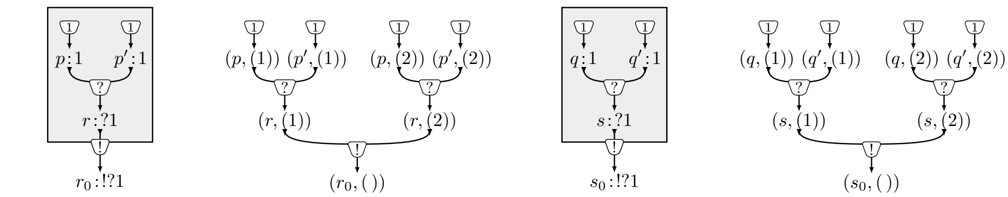
Given a DiLL-ps R, $\rho \in \mathcal{T}_R$, a *box*-cell l of R and a copy with index a (in ρ) of the content of the box of l, Lemma 23.1 says that if in ρ there are at least two copies of the content of the box of l, then the ?-accessible ports in ρ from $(\mathsf{prid}_R(l), a)$ (which is a premise of a !-cell of ρ corresponding to l) are contained in the copy with index a of the content of the box of l; Lemma 23.2 means that if ρ is R-fat and R is box-connected, then the ?-accessible ports in ρ from $(\mathsf{prid}_R(l), a)$ contains all the copy with index a of the content of the box of l. Lemma 23.3 just puts together Lemmas 23.1-23.2. Fig. 6 and 9 give two counterexamples to Lemma 23.3 if one of its hypotheses does not hold.

▶ **Remark 40** (Box-cells preservation). Let $R, S \in \mathbf{PS}_{\mathsf{MELL}}$, $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$ with $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}})\colon \rho \simeq \sigma$. Let $a \in \mathbb{N}^*$ and $l \in \mathcal{C}^{\mathsf{box}}_R$: if $(\mathsf{prid}_R(l), a) \in \mathcal{P}_\rho$ then there are $o \in \mathcal{C}^{\mathsf{box}}_S$ and $b \in$



(a) $R \in \mathbf{PS}_{\mathsf{DiLL}} \smallsetminus \mathbf{PS}_{\mathsf{MELL}}$ **(b)** $\rho \in \mathcal{T}_R$

Figure 10 A box-connected DiLL-ps R (Fig. 10a, with $\mathcal{C}_R^{\mathsf{box}} = \{l\}$ and $\mathcal{C}_R^{!} \smallsetminus \mathcal{C}_R^{\mathsf{box}} = \{o\}$) and a 2-diffnet ρ (Fig. 10b) of R. See also Remark 40.



(a) The MELL-ps R. **(b)** $\rho \in \mathcal{T}_R$. **(c)** The MELL-ps S. **(d)** $\sigma \in \mathcal{T}_S$.

Figure 11 Two box-connected and isomorphic MELL-ps R and S (where $\mathcal{C}_R^{\mathsf{box}} = \{l\}$ and $\mathcal{C}_S^{\mathsf{box}} = \{o\}$), with $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$ strongly fat.

$\mathbb{N}^*$ such that $\varphi_{\mathcal{P}}((\mathsf{prid}_R(l), a)) = (\mathsf{prid}_S(o), b)$ and $\varphi_{\mathcal{C}}((l, a^-)) = (o, b^-)$, as in a MELL-ps !-cells and *box*-cells coincide.[7] Analogously for any $b \in \mathbb{N}^*$ and $o \in \mathcal{C}_S^{\mathsf{box}}$ with $(\mathsf{prid}_S(o), b) \in \mathcal{P}_\sigma$.

Remark 40 is false in general if R or S is a DiLL-ps: for instance, if $R \in \mathbf{PS}_{\mathsf{DiLL}} \smallsetminus \mathbf{PS}_{\mathsf{MELL}}$ and $\rho \in \mathcal{T}_R$ are as in Fig. 10, it is easy to find a $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon \rho \simeq \rho$ with $\varphi_{\mathcal{C}}\big((l, ())\big) = (o, ())$, *i.e.* φ maps the !-cell of ρ corresponding to the *box*-cell of R into the !-cell of ρ not corresponding to the *box*-cell of R. For this reason our main result (Thm. 26) is stated only for MELL-ps.

▶ **Fact 41.** *Let R be a DiLL-ps and $\rho \in \mathcal{T}_R$ with $(p, a), (p', a') \in \mathcal{P}_\rho$. Suppose that, for any $l \in \mathcal{C}_R^{\mathsf{box}}$ and $c \in \mathbb{N}^*$ such that $(\mathsf{prid}_R(l), c) \in \mathcal{P}_\rho$, one has $(p, a) \in \mathcal{P}_\rho^{l,c}$ iff $(p', a') \in \mathcal{P}_\rho^{l,c}$, where $\mathcal{P}_\rho^{l,c} = \{(q, d) \in \mathcal{P}_\rho \mid c \sqsubseteq d,\ q \in \mathsf{inbox}_R(l)\}$. Then, $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p')$ and $a = a'$.*

Proof. By hypothesis, $p \in \mathsf{inbox}_R(l)$ iff $p' \in \mathsf{inbox}_R(l)$ for any $l \in \mathcal{C}_R^{\mathsf{box}}$, hence $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p')$. In particular, if $l = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p')$ (and hence $p, p' \in \mathsf{inbox}_R(l)$ and $|a| = \mathsf{depth}_R(\mathsf{prid}_R(l)) = |a'|$), then by hypothesis we have $c \sqsubseteq a$ iff $c \sqsubseteq a'$ for any $c \in \mathbb{N}^*$ such that $(\mathsf{prid}_R(l), c) \in \mathcal{P}_\rho$, but for such a c one has $|c| = \mathsf{depth}_R(\mathsf{prid}_R(l))$. Therefore, $a = a'$. ◀

▶ **Example 42.** Let R and S be two box-connected and isomorphic MELL-ps as in Fig. 11a,11c, let $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$ as in Fig. 11b,11d. Let $\varphi_{\mathcal{P}} \colon \mathcal{P}_\rho \to \mathcal{P}_\sigma$ be the function defined by $\varphi_{\mathcal{P}}((r_0, ())) = (s_0, ())$ and:

$$\begin{aligned}
\varphi_{\mathcal{P}}((r,(1))) &= (s,(2)) & \varphi_{\mathcal{P}}((r,(2))) &= (s,(1)) & \varphi_{\mathcal{P}}((p,(1))) &= (q',(2)) \\
\varphi_{\mathcal{P}}((p',(1))) &= (q,(2)) & \varphi_{\mathcal{P}}((p,(2))) &= (q,(1)) & \varphi_{\mathcal{P}}((p',(2))) &= (q',(1))\,.
\end{aligned}$$

[7] Recall that $(l, a^-) \in \mathcal{C}_\rho^{!}$ and $(o, b^-) \in \mathcal{C}_\sigma^{!}$ with $(\mathsf{prid}_R(l), a) \in \mathsf{P}_\rho^{\mathsf{aux}}((l, a^-))$ and $(\mathsf{prid}_S(o), b) \in \mathsf{P}_\sigma^{\mathsf{aux}}((o, b^-))$, according to Remark 31.2.

Then, there exists a suitable bijection $\varphi_{\mathcal{C}} \colon \mathcal{C}_\rho \to \mathcal{C}_\sigma$ such that $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon \rho \simeq \sigma$, but $\mathsf{forget}_{\mathcal{P}}^{\rho,R}\big(\varphi_{\mathcal{P}}((p,(l,1)))\big) = q' \neq q = \mathsf{forget}_{\mathcal{P}}^{\rho,R}\big(\varphi_{\mathcal{P}}((p,(l,2)))\big)$. Therefore, the relation $\phi_{\mathcal{P}} \subseteq \mathcal{P}_R \times \mathcal{P}_S$ defined by:

$$(p,q) \in \phi_{\mathcal{P}} \iff \text{for some finite sequences } a,b \text{ one has } \varphi_{\mathcal{P}}((p,a)) = (q,b)$$

is not a function from $\mathcal{P}_R$ to $\mathcal{P}_S$. Similarly, the relation $\phi_{\mathcal{C}} \subseteq \mathcal{C}_R \times \mathcal{C}_S$ defined analogously is not a function from $\mathcal{C}_R$ to $\mathcal{C}_S$.

Example 42 shows that to build an isomorphism between two box-connected MELL-ps starting from two isomorphic strongly fat elements of their (labelled) Taylor expansion is a delicate issue. However, an isomorphism between two strongly fat elements of the (labelled) Taylor expansion has another preservation property, as stated in the next lemma (a corollary of Lemma 23, via Remark 21 and Fact 41), that allows the above issue to be overcome.

▶ **Corollary 24** (Boxes and copies preservation). *Let $R, S \in \mathbf{PS}_{\mathsf{MELL}}$, $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$ with $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon \rho \simeq \sigma$. If R and S are box-connected and ρ and σ are strongly fat, then for any $(p,a), (p',a') \in \mathcal{P}_\rho$ and $(q,b), (q',b') \in \mathcal{P}_\sigma$ with $\varphi_{\mathcal{P}}((p,a)) = (q,b)$ and $\varphi_{\mathcal{P}}((p',a')) = (q',b')$:* Stated at p. 13

1. (copies preserv.) $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p')$ *and* $a = a'$ *iff* $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q')$ *and* $b = b'$*;*
2. (boxes preserv.) $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p')$ *iff* $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q')$*.*

Proof.

1. We prove only the left-to-right direction of the equivalence, the proof of the other direction is analogous, by symmetry. Let us suppose $a = a'$ and $\mathsf{box}^{\mathsf{ext}}_R(p) = \mathsf{box}^{\mathsf{ext}}_R(p')$.
   Let $l \in \mathcal{C}_R^{\mathsf{box}}$ and $c \in \mathbb{N}^*$ be such that $(\mathsf{prid}_R(l), a) \in \mathcal{P}_\rho$ (*i.e.* $c \in \mathsf{f}(l)$, where $\mathsf{f} \in \mathcal{T}_R^{\mathsf{proto}}$ is such that $\rho = \tau_R(\mathsf{f})$). Since $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p')$, one has $p \in \mathsf{inbox}_R(l)$ iff $p' \in \mathsf{inbox}_R(l)$; hence $(p,a) \in \mathcal{P}_\rho^{l,c}$ iff $(p',a) \in \mathcal{P}_\rho^{l,c}$, where we have set $\mathcal{P}_\rho^{l,c} = \{(r, c \cdot c') \in \mathcal{P}_\rho \mid r \in \mathsf{inbox}_R(l) \text{ and } c' \in \mathbb{N}^*\}$. By Lemma 23.3 (since R and S are box-connected, while ρ and σ are strongly fat and hence, respectively, strongly R-fat and strongly S-fat, according to Remark 35.1), $(p,a) \in \mathsf{acces}^?_\rho((\mathsf{prid}_R(l), c))$ iff $(p',a) \in \mathsf{acces}^?_\rho((\mathsf{prid}_R(l), c))$. Since φ is an isomorphism between DiLL_0-ps, then $(q,b) \in \mathsf{acces}^?_\sigma(\varphi_{\mathcal{P}}((\mathsf{prid}_R(l), c)))$ iff $(q',b') \in \mathsf{acces}^?_\sigma(\varphi_{\mathcal{P}}((\mathsf{prid}_R(l), c)))$ by Remark 21. According to Remarks 31.2 and 40, there are $o \in \mathcal{C}_R^{\mathsf{box}}$ and $d \in \mathbb{N}^*$ such that $\varphi_{\mathcal{P}}((\mathsf{prid}_R(l), c)) = (\mathsf{prid}_S(o), d) \in \mathcal{P}_\sigma$ and $\varphi_{\mathcal{C}}((l, c^-)) = (o, d^-) \in \mathcal{C}_\sigma$ with $(\mathsf{prid}_S(o), b) \in \mathsf{P}^{\mathsf{aux}}_\sigma((o, d^-))$. Therefore, $(q,b) \in \mathsf{acces}^?_\sigma((\mathsf{prid}_S(o), d))$ iff $(q',b') \in \mathsf{acces}^?_\sigma((\mathsf{prid}_S(o), d))$. By Lemma 23.3, $(q,b) \in \mathcal{P}_\sigma^{o,d}$ iff $(q',b') \in \mathcal{P}_\sigma^{o,d}$, where we have set $\mathcal{P}_\sigma^{o,d} = \{(r, d \cdot d') \in \mathcal{P}_\sigma \mid r \in \mathsf{inbox}_S(o) \text{ and } d' \in \mathbb{N}^*\}$.
   Since Remark 40 establishes a one-to-one correspondence between $c \in \mathbb{N}^*$ and $l \in \mathcal{C}_R^{\mathsf{box}}$ such that $(\mathsf{prid}_R(l), c) \in \mathcal{P}_\rho$, and $d \in \mathbb{N}^*$ and $o \in \mathcal{C}_S^{\mathsf{box}}$ such that $(\mathsf{prid}_S(o), d) \in \mathcal{P}_\sigma$, we can apply Fact 41 and hence $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q')$ and $b = b'$.
2. According to Remark 32.2, since $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p')$ and $(p,a) \in \mathcal{P}_\rho$, then $(p',a) \in \mathcal{P}_\rho$. Thus, by Corollary 24.1, one has $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q')$. ◀

▶ **Lemma 25** (Building isomorphism). *Let $R, S \in \mathbf{PS}_{\mathsf{MELL}}$, $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$. Suppose ρ and σ are strongly fat and canonical, and $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon \rho \simeq \sigma$. Let $\phi_{\mathcal{P}} \colon \mathcal{P}_R \to \mathcal{P}_S$ and $\phi_{\mathcal{C}} \colon \mathcal{C}_R \to \mathcal{C}_S$ be functions defined in Eq.* (2)*. If R and S are box-connected, then $\phi = (\phi_{\mathcal{P}}, \phi_{\mathcal{C}}) \colon R \simeq S$.* Stated at p. 14

$$\begin{aligned} \phi_{\mathcal{P}}(p) &= \mathsf{forget}_{\mathcal{P}}^{\sigma,S}(\varphi_{\mathcal{P}}((p,a))) \quad \textit{for every } p \in \mathcal{P}_R \textit{ where } (p,a) \in \mathcal{P}_\rho \textit{ with } a \in \{1\}^*; \\ \phi_{\mathcal{C}}(l) &= \mathsf{forget}_{\mathcal{C}}^{\sigma,S}(\varphi_{\mathcal{C}}((l,a))) \quad \textit{for every } l \in \mathcal{C}_R \textit{ where } (l,a) \in \mathcal{C}_\rho \textit{ with } a \in \{1\}^*. \end{aligned} \tag{2}$$

Proof. First, observe that ρ is strongly R-fat and S is strongly S-fat according to Remark 35.1. Besides the functions $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{C}}$ are well-defined: indeed, by Remark 35.2, for every $p \in \mathcal{P}_\rho$

there is a $a \in \{1\}^*$ such that $(p, a) \in \mathcal{P}_\rho$, since ρ is canonical; according to the depth compatibility condition of Def. 12, such a a is unique.

We prove that $\phi_\mathcal{P} \colon \mathcal{P}_R \to \mathcal{P}_S$ is bijective. The proof that $\phi_\mathcal{C} \colon \mathcal{C}_R \to \mathcal{C}_S$ is bijective is perfectly analogous and it is left to the reader.

Injectivity: Let $p, p' \in \mathcal{P}_R$ with $p \neq p'$. Then, for the unique $a, a' \in \{1\}^*$ such that $(p, a), (p', a') \in \mathcal{P}_\rho$, one has $(p, a) \neq (p', a')$. Let $\varphi_\mathcal{P}(p, a) = (q, b) \in \mathcal{P}_\sigma$ and $\varphi_\mathcal{P}(p', a') = (q', b') \in \mathcal{P}_\sigma$: by definition of ϕ, $\phi_\mathcal{P}(p) = q$ and $\phi_\mathcal{P}(p') = q'$. Since $\varphi_\mathcal{P}$ is injective, $(q, b) \neq (q', b')$. There are only two cases:

- either $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p')$ and hence $\mathsf{depth}_R(p) = \mathsf{depth}_R(p')$. By the depth compatibility condition of Def. 12, one has $|a| = |a'|$. Hence, from $a, a' \in \{1\}^*$ it follows that $a = a'$. According to copies preservation (Corollary 24.1), $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q')$ and $b = b'$. As $(q, b) \neq (q', b')$, then $q \neq q'$;
- or $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) \neq \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p')$ and hence $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q) \neq \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q')$ according to boxes preservation (Corollary 24.2); so, $q \neq q'$.

In both cases, $\phi_\mathcal{P}(p) = q \neq q' = \phi_\mathcal{P}(p')$. Therefore, ϕ is injective.

Surjectivity: Let $q \in \mathcal{P}_S$ and $\mathcal{P}^q_\sigma = \{(q, b) \in \mathcal{P}_\sigma \mid b \in \mathbb{N}^*\} = \{(q, b_1), \dots, (q, b_n)\}$ for some $n \in \mathbb{N}$ and some pairwise distinct $b_1, \dots, b_n \in \mathbb{N}^*$. By Remark 35.2, $n > 0$. Since $\varphi_\mathcal{P}$ is bijective, $\overline{\varphi_\mathcal{P}}^{-1}(\mathcal{P}^q_\sigma) = \{(p_1, a_1), \dots, (p_n, a_n)\} \subseteq \mathcal{P}_\rho$ where $\varphi_\mathcal{P}((p_i, a_i)) = (q, b_i)$ for all $1 \leq i \leq n$, and $p_1, \dots, p_n \in \mathcal{P}_R$ and $a_1, \dots a_n \in \mathbb{N}^*$. By boxes preservation (Corollary 24.2) $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p_i) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p_j)$ for any $1 \leq i, j \leq n$. By copies preservation (Corollary 24.1), $a_1, \dots, a_n$ are pairwise distinct since $b_1, \dots, b_n$ are so. Therefore, given $\mathsf{f} \in \mathcal{T}^{\mathsf{proto}}_R$ such that $\rho = \tau_R(\mathsf{f})$, one has $\mathsf{f}(\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p_i)) = \{a_1, \dots, a_n\}$ for any $1 \leq i \leq n$. Since ρ is canonical, there exists $1 \leq i \leq n$ such that $a_i \in \{1\}^*$: thus, for such a i, $\phi_\mathcal{P}(p_i) = \mathsf{forget}^{\rho,R}_\mathcal{P}(\varphi_\mathcal{P}((p_i, a_i))) = q$. Hence, ϕ is surjective.

We now prove that $(\phi_\mathcal{P}, \phi_\mathcal{C}) \colon G_R \simeq G_S$, that is $\phi_\mathcal{P}$ and $\phi_\mathcal{C}$ make diagrams in Fig. 2a (see Def. 5) commute. Let $\mathsf{f} \in \mathcal{T}^{\mathsf{proto}}_R$ be such that $\rho = \tau_R(\mathsf{f})$. By Remark 35.4, there is a pruning of f_1 of f such that $\tau_R(\mathsf{f}_1)$ is a 1-diffnet of R. According to Remark 14, we can suppose without loss of generality that f_1 is canonical. Let $\rho_1 = \tau_R(\mathsf{f}_1)$. Note that $\mathcal{P}_{\rho_1} \subseteq \mathcal{P}_\rho$ and $\mathcal{C}_{\rho_1} \subseteq \mathcal{C}_\rho$, therefore we can consider the images $\overline{\varphi_\mathcal{P}}(\mathcal{P}_{\rho_1})$ and $\overline{\varphi_\mathcal{C}}(\mathcal{C}_{\rho_1})$ of $\mathcal{P}_{\rho_1}$ and $\mathcal{C}_{\rho_1}$ via $\varphi_\mathcal{P}$ and $\varphi_\mathcal{C}$, respectively. By copies preservation (Corollary 24.1), given $(q, b) \in \overline{\varphi_\mathcal{P}}(\mathcal{P}_{\rho_1})$, for any $(q', b) \in \mathcal{P}_\sigma$ such that $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q) = \mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q')$ one has $(q', b) \in \overline{\varphi_\mathcal{P}}(\mathcal{P}_{\rho_1})$ (and analogously for cells): the whole b copy of the content of the box of $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q)$ is in $\overline{\varphi_\mathcal{P}}(\mathcal{P}_{\rho_1})$. This means that $\overline{\varphi_\mathcal{P}}(\mathcal{P}_{\rho_1})$ and $\overline{\varphi_\mathcal{C}}(\mathcal{C}_{\rho_1})$ are generated by some $\mathsf{g} \in \mathcal{T}^{\mathsf{proto}}_S$ as in Def. 13: in other words, if we set $\sigma_1 = \tau_S(\mathsf{g})$ and $\mathsf{h} \in \mathcal{T}^{\mathsf{proto}}_S$ is such that $\sigma = \tau_S(\mathsf{h})$, g is a pruning (not necessarily canonical) of h such that $(\varphi_\mathcal{P}\restriction_{\mathcal{P}_{\rho_1}}, \varphi_\mathcal{C}\restriction_{\mathcal{C}_{\rho_1}}) \colon \rho_1 \simeq \sigma_1$ where $\sigma_1 \in \mathcal{T}_S$ is a 1-diffnet of S (since ρ_1 is a 1-diffnet of R). Since ρ_1 is canonical, one has $\phi_\mathcal{P} = \mathsf{forget}^{\rho,R}_\mathcal{P} \circ \varphi_\mathcal{P}\restriction_{\mathcal{P}_{\rho_1}}$ and $\phi_\mathcal{C} = \mathsf{forget}^{\rho,R}_\mathcal{C} \circ \varphi_\mathcal{C}\restriction_{\mathcal{C}_{\rho_1}}$, therefore $(\phi_\mathcal{P}, \phi_\mathcal{C}) \colon G_R \simeq G_S$ according to Fact 19.2.

Following Def. 5, we now prove that $\mathsf{im}(\phi_\mathcal{C}\restriction_{\mathcal{C}^{\mathsf{box}}_R}) = \mathcal{C}^{\mathsf{box}}_S$ and $\mathsf{im}(\varphi_\mathcal{P}\restriction_{\mathcal{D}oors_R}) = \mathcal{D}oors_S$. The first identity follows immediately from Remark 40. The second identity follows from Corollary 24.2, since $\mathcal{D}oors_R$ and $\mathcal{D}oors_S$ are entirely characterized by $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}$ and $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}$ respectively, according to Remark 29.6 (for $\mathcal{D}oors^{cut}_R$ and $\mathcal{D}oors^{cut}_S$) and Remarks 29.3 and 30.1 (for $\mathcal{D}oors^!_R \cup \mathcal{D}oors^?_R$ and $\mathcal{D}oors^!_S \cup \mathcal{D}oors^?_S$).

We complete the proof that $(\phi_\mathcal{P}, \phi_\mathcal{C}) \colon R \simeq S$ by showing that $\phi_\mathcal{P}$ and $\phi_\mathcal{C}$ make diagram in Fig. 2b (Def. 5) commute. This fact follows from Corollary 24.2, since $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_R}(p) = \mathsf{box}_R(p)$ and $\mathsf{box}^{\mathsf{ext}}_{\mathcal{P}_S}(q) = \mathsf{box}_S(q)$ for any $p \in \mathcal{D}oors_R$ and $q \in \mathcal{D}oors_S$ according to Remark 29.2. ◀

A.5 Omitted proofs of Section 5

▶ **Corollary 27** (Injectivity for box-connected MELL). *Let R and S be cut-free, η-expanded and box-connected* MELL*-ps with conclusions of the same type. If $[\![R]\!] = [\![S]\!]$, then $R \simeq S$.* Stated at p. 14

Proof. Let Γ be the type of the conclusions of R and S. From $[\![R]\!] = [\![S]\!]$ it follows that $[\![R]\!]_{\mathrm{inj}}/\!\sim_{\wp\Gamma} = [\![S]\!]_{\mathrm{inj}}/\!\sim_{\wp\Gamma}$ (see Def. 61-62 in Appendix C for the meaning of $[\![R]\!]_{\mathrm{inj}}/\!\sim_{\wp\Gamma}$). By Prop. 64 (see Appendix C), since R and S are cut-free and η-expanded, one has $\mathcal{T}_R^{\simeq} = \mathcal{T}_S^{\simeq}$, in particular they have the same element of order 2 (which is strongly fat). As R and S are box-connected, from Thm. 26 it follows that $R \simeq S$. ◀

B Computing boxes

Unlike usual syntaxes of LL- or DiLL-proof structures (see for example [12, 15, 5, 21]), in our syntax there is no explicit (inductive) constructor for boxes: a box in a pps Φ is defined as a particular sub(hyper)graph of Φ. This more "geometric" approach was followed for example in [3, 20, 16, 6]. In our syntax, the boxes in a DiLL-ps R are reconstructed in a non-inductive way using some "geometric" informations coming from R. Roughly speaking, given a *box*-cell l of R, the box associated with l defined) is all that is above (in the sense of $\leq_R$) of l and the ports in $\mathsf{box}_R^{-1}(l')$ for any *box*-cell l' above l (in the sense of $\leq_{\mathcal{C}_R^{\mathsf{box}}}$). In this subsection we show how to compute all that. It is worth noting that *this section is not necessary to prove our main results*: our goal here is to convince the reader that in a DiLL-ps R we have all the information to recover the boxes of R, even if there is not an explicit constructor for them.

We start with a lemma which holds for pps.

▶ **Lemma 43** (About minimal elements for $\mathsf{box}_\Phi^{\mathsf{ext}}$). *Let Φ be a pps and $p \in \mathcal{P}_\Phi$. If, for all $q \in \mathcal{P}_\Phi$, $q <_\Phi p$ implies $\mathsf{box}_\Phi^{\mathsf{ext}}(q) <_{\mathcal{C}_\Phi^{\mathsf{box}} \cup \{\bullet\}} \mathsf{box}_\Phi^{\mathsf{ext}}(p)$, then $p \in \mathcal{D}oors_\Phi$.*

Proof. According to Definition 8 and Remark 29.2, either $\mathsf{box}_\Phi^{\mathsf{ext}}(p) = \bullet$ or there is $q \in \downarrow_\Phi p \cap \mathcal{D}oors_\Phi$ such that $\mathsf{box}_\Phi^{\mathsf{ext}}(p) = \mathsf{box}_\Phi(q) = \mathsf{box}_\Phi^{\mathsf{ext}}(q)$. It follows from the hypothesis that $q \not<_\Phi p$, but $q \leq_\Phi p$, therefore $q = p$. ◀

The converse of Lemma 43 does not hold in general for pps: in the pps Ψ_1 (Figure 1b), $p \in \mathcal{D}oors_{\Psi_1}$ but $\mathsf{box}_{\Psi_1}^{\mathsf{ext}}(p) = \mathsf{box}_{\Psi_1}^{\mathsf{ext}}(q)$ with $q <_{\Psi_1} p$.

The following fact gives a characterization of $\leq_{\mathcal{C}_\Phi^{\mathsf{box}}}$.

▶ **Fact 44** (Equivalent definition of $\leq_{\mathcal{C}_\Phi^{\mathsf{box}}}$). *Let Φ be a pps and $l, l' \in \mathcal{C}_\Phi^{\mathsf{box}}$: $l \leq_{\mathcal{C}_\Phi^{\mathsf{box}}} l'$ iff there are $p, p' \in \mathcal{P}_\Phi$ such that $p \leq_\Phi p'$ and $\mathsf{box}_{\mathcal{P}_\Phi}^{\mathsf{ext}}(p) = l$ and $\mathsf{box}_{\mathcal{P}_\Phi}^{\mathsf{ext}}(p') = l'$.*

Proof. $\Leftarrow$: According to Definition 8, there are $q = \max_{\leq_{\mathcal{C}_\Phi^{\mathsf{box}}}}(\downarrow_{\mathcal{P}_\Phi} p \cap \mathcal{D}oors_\Phi)$ and $q' = \max_{\leq_{\mathcal{C}_\Phi^{\mathsf{box}}}}(\downarrow_{\mathcal{P}_\Phi} p' \cap \mathcal{D}oors_\Phi)$ with $l = \mathsf{box}_\Phi^{\mathsf{ext}}(p) = \mathsf{box}_\Phi(q)$ and $l' = \mathsf{box}_\Phi^{\mathsf{ext}}(p') = \mathsf{box}_\Phi(q')$; in particular, $q \leq_\Phi p$ and $q' \leq_\Phi p'$. As $p \leq_\Phi p'$, one has $q \leq_\Phi p'$. Since $\downarrow_{\mathcal{P}_\Phi} p'$ is finite, it is totally ordered by $\leq_\Phi$ according to Fact 4, so either $q \leq_\Phi q'$ or $q' <_\Phi q$. But $q' \not<_\Phi q$ because of the maximality of q', since $q \in \downarrow_{\mathcal{P}_\Phi} p' \cap \mathcal{D}oors_\Phi$. Thus, $q \leq_\Phi q'$ and therefore $l = \mathsf{box}_\Phi(q) \leq_{\mathcal{C}_\Phi^{\mathsf{box}}} \mathsf{box}_\Phi(q') = l'$ according to Definition 9.

$\Rightarrow$: According to Definition 9 and Remark 29.2, there are $p, p' \in \mathcal{D}oors_\Phi$ such that $p \leq_\Phi p'$ and $l = \mathsf{box}_\Phi(p) = \mathsf{box}_\Phi^{\mathsf{ext}}(p)$ and $l' = \mathsf{box}_\Phi(p') = \mathsf{box}_\Phi^{\mathsf{ext}}(p)$. ◀

From now on, we will consider DiLL-ps only.

▶ **Definition 45** (Doors of a box). Let R be a DiLL-ps and $l \in \mathcal{C}_R^{\mathsf{box}}$. We set:

- $\mathsf{auxd}_R(l) = \{q \in \mathcal{D}oors^?_R \mid l \leq_{\mathcal{C}^{\mathsf{box}}_R} \mathsf{box}_R(q)$ and $\mathsf{box}^{\mathsf{ext}}_R(\mathsf{pred}_R(q)) <_{\mathcal{C}^{\mathsf{box}}_R \cup \{\bullet\}} l\}$, whose elements are the *aux-doors of the box of l (in R)*,
- $\mathsf{cutd}_R(l) = \{q \in \mathcal{D}oors^{cut}_R \mid l \leq_{\mathcal{C}^{\mathsf{box}}_R} \mathsf{box}_R(q)\}$, whose elements are the *cut-doors of the box of l (in R)*,
- $\mathsf{doors}_R(l) = \{\mathsf{prid}_R(l)\} \cup \mathsf{auxd}_R(l) \cup \mathsf{cutd}_R(l)$, whose elements are the *doors of the box of l (in R)*,

Intuitively, given a *box*-cell l in a DiLL-ps R, the doors of the box of l mark the boundary of the box of l, and the ports in $\mathsf{inbox}_R(l)$, which represent the content of the box of l, are all and only the ports above (in the sense of $\leq_R$) the doors of the box of l, as explained in Fact 47.2 below.

▶ **Remark 46.** It follows from Definition 45 and Remarks 29.1-2 that, given a DiLL-ps R and $l \in \mathcal{C}^{\mathsf{box}}_R$, if $p \in \mathsf{doors}_R(l)$ then $p \in \mathcal{D}oors_R$ and $l \leq_{\mathcal{C}^{\mathsf{box}}_R} \mathsf{box}^{\mathsf{ext}}_R(p) \neq \bullet$.

Since we use generalized ?-cells, the same premise of a ?-cell might be an *aux*-door of several boxes of *box*-cells, *i.e.* in a DiLL-ps R there might be $p \in \mathcal{D}oors^?_R$ and $l, l' \in \mathcal{C}^{\mathsf{box}}_R$ such that $p \in \mathsf{auxd}_R(l) \cap \mathsf{auxd}_R(l')$ and $l \neq l'$. Analogously, the same premise of a *cut*-cell might be a *cut*-door of several boxes of *box*-cells.

Fact 47.1 below gives a criterion to find the doors of the box of a *box*-cell.

▶ **Fact 47** (Geometric description of the content of a box). *Let R be a* DiLL*-ps.*

1. $\bigcup_{l \in \mathcal{C}^{\mathsf{box}}_R} \mathsf{doors}_R(l) = \mathcal{D}oors_R$.
2. *For any $l \in \mathcal{C}^{\mathsf{box}}_R$, one has $\mathsf{inbox}_R(l) = \{q \in \mathcal{P}_R \mid \exists p \in \mathsf{doors}_R(l) : p \leq_R q\}$.*
3. *For all $p, q \in \mathcal{P}_R$ and $l \in \mathcal{C}^{\mathsf{box}}_R$, if $p <_R q$ and $q \in \mathsf{doors}_R(l)$, then $p \notin \mathsf{inbox}_R(l)$; in particular, $p \notin \mathsf{doors}_R(l)$.*

So, to compute the content of the box associated with a *box*-cell l of a DiLL-ps R, first one has to identify the doors of the box of l and then one has to take all and only the ports above (in the sense of $\leq_R$) such doors (Fact 47.2). Moreover, Fact 47.3 says that two doors of the box of l cannot be above each other and hence all the ports below a door of the box of l are outside the box of l. Facts 47.2-3 mean that the minimal elements (with respect to $\leq_{\mathcal{C}^{\mathsf{box}}_R}$) of $\mathsf{inbox}_R(l)$ are the doors of the box of l, in particular the premise of l is the unique such minimal element that is the premise of a !-cell, all other such minimal elements being premises of *cut*-cells or ?-cells. More precisely, condition 3 (resp. 4) in Definition 10 plays a crucial role in the proof that among such minimal elements there is at least (resp. at most) one premise of a !-cell.

It is worth noting that Fact 47.2 does not imply Fact 47.3.

▶ **Proposition 48** (Nesting condition). *Let R be a* DiLL*-ps and $l, l' \in \mathcal{C}^{\mathsf{box}}_R$.*

1. *If $l <_{\mathcal{C}^{\mathsf{box}}_R} l'$ then $\mathsf{inbox}_R(l') \subsetneq \mathsf{inbox}_R(l)$.*
2. *If $l \not\leq_{\mathcal{C}^{\mathsf{box}}_R} l'$ and $l' \not\leq_{\mathcal{C}^{\mathsf{box}}_R} l$, then $\mathsf{inbox}_R(l) \cap \mathsf{inbox}_R(l') = \emptyset$.*
3. *If $l \neq l'$ then either $\mathsf{inbox}_R(l) \cap \mathsf{inbox}_R(l') = \emptyset$, or $\mathsf{inbox}_R(l) \subsetneq \mathsf{inbox}_R(l')$ or $\mathsf{inbox}_R(l') \subsetneq \mathsf{inbox}_R(l)$.*

Proof. 1. First, we show that $\mathsf{inbox}_R(l) \neq \mathsf{inbox}_R(l')$. Let $p = \mathsf{prid}_R(l)$ and $p' = \mathsf{prid}_R(l')$. By Remark 29.2, $\mathsf{box}^{\mathsf{ext}}_R(p) = l$ and $l' = \mathsf{box}^{\mathsf{ext}}_R(p')$, hence $p \in \mathsf{inbox}_R(l)$ and $p' \in \mathsf{inbox}_R(l')$. One has $\{p, p'\} \not\subseteq \mathsf{inbox}_R(l) \cap \mathsf{inbox}_R(l')$, otherwise $l \leq_{\mathcal{C}^{\mathsf{box}}_R} \mathsf{box}^{\mathsf{ext}}_R(p') = l'$ and $l' \leq_{\mathcal{C}^{\mathsf{box}}_R} \mathsf{box}^{\mathsf{ext}}_R(p) = l$ according to Definition 45, and hence $l = l'$ by condition 1 in Definition 10, but this is impossible since $l \neq l'$. Therefore, either $p \in \mathsf{inbox}_R(l) \smallsetminus \mathsf{inbox}_R(l')$ or $p' \in \mathsf{inbox}_R(l') \smallsetminus \mathsf{inbox}_R(l)$: this proves that $\mathsf{inbox}_R(l) \neq \mathsf{inbox}_R(l')$.

We now show that $\mathsf{inbox}_R(l') \subseteq \mathsf{inbox}_R(l)$. If $q \in \mathsf{inbox}_R(l')$ then, according to Definition 45, $l' \leq_{\mathcal{C}_R^{\mathsf{box}}} \mathsf{box}_R^{\mathsf{ext}}(q)$. By hypothesis, $l \leq_{\mathcal{C}_R^{\mathsf{box}}} l'$. Therefore, $l \leq_{\mathcal{C}_R^{\mathsf{box}}} \mathsf{box}_R^{\mathsf{ext}}(q)$ and thus $q \in \mathsf{inbox}_R(l)$.

2. If $p \in \mathsf{inbox}_R(l) \cap \mathsf{inbox}_R(l')$ then, by Definition 45, $p \in \mathsf{dom}(\mathsf{box}_R^{\mathsf{ext}})$ and $l, l' \leq_{\mathcal{C}_R^{\mathsf{box}}} \mathsf{box}_R^{\mathsf{ext}}(p)$. By condition 1 in Definition 10 and since $\mathcal{C}_R^{\mathsf{box}}$ is finite, $\downarrow_{\mathcal{C}_R^{\mathsf{box}}} \mathsf{box}_R^{\mathsf{ext}}(q)$ is totally ordered by $\leq_{\mathcal{C}_R^{\mathsf{box}}}$, so either $l \leq_{\mathcal{C}_R^{\mathsf{box}}} l'$ or $l' \leq_{\mathcal{C}_R^{\mathsf{box}}} l$.
3. Suppose that $\mathsf{inbox}_R(l) \cap \mathsf{inbox}_R(l') \neq \emptyset$. By Proposition 48.2, either $l <_{\mathcal{C}_R^{\mathsf{box}}} l'$ or $l' <_{\mathcal{C}_R^{\mathsf{box}}} l$ (because $l \neq l'$ by hypothesis). By Proposition 48.1, either $\mathsf{inbox}_R(l') \subsetneq \mathsf{inbox}_R(l)$ or $\mathsf{inbox}_R(l) \subsetneq \mathsf{inbox}_R(l')$. ◀

Now we are able to define the box associated with every *box*-cell of a DiLL-ps and prove that it is a DiLL-ps (Proposition 50).

▶ **Definition 49** (Box of a *box*-cell)**.** Let R be a DiLL-ps and $l \in \mathcal{C}_R^{\mathsf{box}}$. The *box of* l *in* R is the 9-tuple $R_l = (\mathcal{P}_{R_l}, \mathcal{C}_{R_l}, \mathsf{tc}_{R_l}, \mathsf{P}_{R_l}^{\mathsf{pri}}, \mathsf{P}_{R_l}^{\mathsf{aux}}, \mathsf{P}_{R_l}^{\mathsf{left}}, \mathsf{tp}_{R_l}, \mathcal{C}_{R_l}^{\mathsf{box}}, \mathsf{box}_{R_l})$ where:

- $\mathcal{P}_{R_l} = \mathsf{inbox}_R(l) \cup \bigcup \overline{\mathsf{P}_R^{\mathsf{pri}}}(\mathcal{C}_R^{\mathsf{bord}})$;
- $\mathcal{C}_{R_l} = \{l' \in \mathcal{C}_R \mid \big(\mathsf{P}_R^{\mathsf{pri}}(l') \cup \mathsf{P}_R^{\mathsf{aux}}(l')\big) \cap \mathsf{inbox}_R(l) \neq \emptyset\}$ and $\mathsf{tc}_{R_l} = \mathsf{tc}_R\restriction_{\mathcal{C}_{R_l}}$;
- $\mathsf{P}_{R_l}^{\mathsf{pri}} = \mathsf{P}_R^{\mathsf{pri}}\restriction_{\mathcal{C}_{R_l}}$, $\mathsf{P}_{R_l}^{\mathsf{aux}}(l') = \mathsf{P}_R^{\mathsf{aux}}(l') \cap \mathcal{P}_{R_l}$ for all $l' \in \mathcal{C}_{R_l}$, $\mathsf{P}_{R_l}^{\mathsf{left}} = \mathsf{P}_R^{\mathsf{left}}\restriction_{\mathcal{C}_{R_l}^{\otimes,\parr}}$ and $\mathsf{tp}_{R_l} = \mathsf{tp}_R\restriction_{\mathcal{P}_{R_l}}$;
- $\mathcal{C}_{R_l}^{\mathsf{box}} = \mathcal{C}_R^{\mathsf{box}} \cap \mathcal{C}_{R_l}$ and $\mathsf{box}_{R_l} = \mathsf{box}_R\restriction_{\mathsf{inbox}_R(l)}$.

▶ **Proposition 50.** *Let R be a* DiLL*-ps (resp.* MELL*-ps). For every $l \in \mathcal{C}_R^{\mathsf{box}}$, the box R_l of l in R is a* DiLL*-ps (resp.* MELL*-ps), with* $\mathcal{P}_{R_l}^{\mathsf{free}} = \overline{\mathsf{pred}}_R(\{\mathsf{prid}_R(l)\} \cup \mathsf{auxd}_R(l)) \subseteq \overline{\mathsf{P}_R^{\mathsf{pri}}}(\mathcal{C}_R^{\mathsf{bord}})$.

Proof. Left to the reader. ◀

C Relational semantics

Relational Experiments The relational model is the simplest and maybe the most canonical model of LL It can be seen as a degenerate case of Girard's coherent semantics [12], and as such, fomulæ are interpreted as sets and proofs as relations between them.

▶ **Definition 51** (Web of a MELL formula)**.** Let $\mathcal{A}t$ be a countably infinite set such that $\mathcal{A}t \cap (\mathcal{L}_{\mathsf{MELL}} \cup \{()\}) = \emptyset$; the elements of $\mathcal{A}t$ are called *atoms*. By induction, we define a function $|\cdot|$ on MELL formulæ by, for $A \in \mathcal{F}_{\mathsf{MELL}}$:

$$|X^\perp| = |X| = \mathcal{A}t, \text{ for any } X \in \mathcal{V}_{\mathsf{MELL}}; \qquad |1| = |\perp| = \{()\};$$
$$|A \otimes B| = |A \parr B| = |A| \times |B|; \qquad |!A| = |?A| = \mathscr{M}_{\mathrm{fin}}(|A|),$$

For a formula A, the set $|A|$ is called the *web of* A, whose elements are the *points of* A.

So $|A^\perp| = |A|$ for any $A \in \mathcal{F}_{\mathsf{MELL}}$: relational semantics is a degenerate model of MELL.
Following the interpretation of MELL proofs (in sequent calculus) in [1, Appendix] and the notion of multiplicative experiment [12, Definition 3.17], we first define experiments on DiLL_0-ps.

▶ **Definition 52** (Experiment of a DiLL_0-ps)**.** Let Φ be a DiLL_0-ps.
An *experiment* e of Φ is a function associating with every $p \in \mathcal{P}_\Phi$ an element of $|\mathsf{tp}_\Phi(p)|$ verifying the following conditions:

- if $l \in \mathcal{C}_\Phi^{ax}$ with $\mathsf{P}_\Phi^{\mathsf{pri}}(l) = \{p, q\}$, then $\mathsf{e}(p) = \mathsf{e}(q)$;

- if $l \in \mathcal{C}_{\Phi}^{cut}$ with $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p, q\}$, then $\mathsf{e}(p) = \mathsf{e}(q)$;
- if $l \in \mathcal{C}_{\Phi}^{1,\bot}$ with $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l) = \{q\}$, then $\mathsf{e}(q) = ()$;
- if $l \in \mathcal{C}_{\Phi}^{\otimes,\parr}$ with $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_1, p_2\}$, $\mathsf{P}_{\Phi}^{\mathsf{left}}(l) = \{p_1\}$, $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l) = \{q\}$ and $\mathsf{e}(p_i) = a_i$ for any $i \in \{1, 2\}$, then $\mathsf{e}(q) = (a_1, a_2)$;
- if $l \in \mathcal{C}_{\Phi}^{!,?}$, $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_1, \dots, p_n\}$ for some $n \in \mathbb{N}$ and $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l) = \{q\}$, then

$$\mathsf{e}(q) = \sum_{i=1}^{n} [\mathsf{e}(p_i)] .$$

A *partial experiment* is a partial function satisfying the same conditions.

Experiments are well-defined on equivalence classes of DiLL_0-proof structures. Precisely,

▶ **Fact 53.** *Let Φ be a DiLL_0-ps, and e an experiment of Φ. Let Ψ be a DiLL_0-ps so that $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) : \Phi \simeq \Psi$. The isomorphism φ transports naturally e to an experiment of Ψ, noted $\varphi_*\mathsf{e}$, defined by, for all $p \in \mathcal{P}_{\Psi}$, $\varphi_*\mathsf{e}(p) = \mathsf{e}(\varphi_{\mathcal{P}}^{-1}(p))$.*

▶ **Remark 54.** As a consequence of Fact 53, an isomorphism $\varphi \colon \Phi \simeq \Psi$ of DiLL_0-ps induces a bijection φ_* between the set of experiments of Φ and the set of experiments of Ψ.

Relational Semantics In order to define the interpretation of a DiLL_0-ps Φ in the relational model, we have to fix an order on the conclusions of Φ, engendering indexed DiLL_0-ps. The notion of isomorphism extends to indexed DiLL_0-ps: an isomorphism between two indexed DiLL_0-ps is an isomorphism between DiLL_0-ps preserving the order of the conclusions.

▶ **Definition 55** (Indexed DiLL_0-proof structure)**.** An *indexed* DiLL_0*-ps* is a pair $R = (\Phi_R, \mathsf{concl}_R)$ where Φ_R is a DiLL_0-ps and $\mathsf{concl}_R \colon \{1, \dots, n\} \to \mathcal{P}_{\Phi_R}^{\mathsf{free}}$ is a bijection (for $n \in \mathbb{N}$).

The *type of the conclusions of* R is the tuple $(A_1, \dots, A_n)$ where $n = \mathsf{card}(\mathcal{P}_{\Phi_R}^{\mathsf{free}})$ and $\mathsf{tp}_{\Phi_R}(\mathsf{concl}_R(i)) = A_i$ for any $1 \le i \le n$.

The set of indexed DiLL_0-ps is denoted by $\mathbf{PS}_{\mathsf{DiLL}_0}^{\mathsf{ind}}$.

▶ **Definition 56** (Result of an experiment and interpretation of an indexed DiLL_0-ps)**.** Let $R = (\Phi_R, \mathsf{concl}_R)$ be an indexed DiLL_0-ps with $\mathsf{card}(\mathcal{P}_{\Phi_R}^{\mathsf{free}}) = n \in \mathbb{N}$.

Let e be an experiment of Φ_R: the *result of* e *with respect to* concl_R is

$$|\mathsf{e}|_{\mathsf{concl}_R} = (\mathsf{e}(\mathsf{concl}_R(1)), \dots, \mathsf{e}(\mathsf{concl}_R(n))).$$

The *relational interpretation of* R is: $[\![R]\!] = \{|\mathsf{e}|_{\mathsf{concl}_R} \mid \mathsf{e} \text{ is an experiment of } \Phi_R\}$.

The relational interpretation is actually defined on equivalence classes of indexed DiLL_0-ps.

A crucial property of the relational interpretation of indexed DiLL_0-ps introduced in Definition 56 is that it is invariant under cut-elimination [1]. It is also possible to check that it is invariant under η-expansion too.

On a cut-free indexed DiLL_0-ps, all the information of an experiment is in its conclusion: two experiments have the same conclusion if and only if they are equal.

Semantics of DiLL The Taylor expansion associates a set of DiLL_0-ps with a DiLL-ps. We have defined an interpretation of indexed DiLL_0-ps, the relational interpretation $[\![\cdot]\!]$. In order to extend these interpretations to the whole of $\mathbf{PS}_{\mathsf{DiLL}}$, we need to extend Definition 55 to the context of DiLL-ps (which is straightforward) and to remark that an indexed DiLL-ps induces an indexation on the elements of its Taylor expansion.

▶ **Fact 57.** *Let* $R = (\Phi_R, \mathsf{concl}_R)$ *be an indexed* DiLL*-ps. Let* ρ *be a* DiLL_0*-ps in the Taylor expansion of* Φ_R.

The forgetful function $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ *on ports from* ρ *to* R *induces a bijection* $\mathsf{forget}_{\mathsf{concl}}^{\rho,R}$ *between conclusions of* ρ *and conclusions of* R.

The pair $(\rho, \left(\mathsf{forget}_{\mathsf{concl}}^{\rho,R}\right)^{-1} \circ \mathsf{concl}_R)$ *is an indexed* DiLL_0*-ps.*

We still write $\mathcal{T}\colon \mathbf{PS}_{\mathsf{DiLL}}^{\mathsf{ind}} \to \mathscr{P}(\mathbf{PS}_{\mathsf{DiLL}_0}^{\mathsf{ind}})$ the function that maps an indexed DiLL-ps R to its indexed Taylor expansion, that is, the set of indexed DiLL_0-ps whose underlying DiLL_0-ps is in the Taylor expansion of the underlying DiLL-ps of R endowed with the indexation of R, and $\mathcal{T}^{\simeq}$ the quotiented Taylor expansion function.

We define the interpretation of an indexed DiLL-ps through the interpretation of its Taylor expansion.

▶ **Definition 58** (Interpretation of a DiLL-ps)**.** The *relational interpretation* (or *semantics*) of an indexed DiLL-ps R is the set $[\![R]\!] = \bigcup_{\overline{\rho}\in\mathcal{T}^{\simeq}_R}[\![\rho]\!]$.

▶ **Remark 59** (Experiment of a DiLL-proof structure)**.** The definition of an experiment for MLL-proof structures is very straightforward [12]. The presence of exponentials in the full MELL engenders complications when tackled directly (see for instance [20, 6]).

The Taylor expansion acts as a bridge in our definition, hiding all the complexity of the definition of experiments. It allows to retain the simplicity of the MLL framework while being usable for the whole of DiLL.

Injective Semantics The relational interpretation of an indexed DiLL_0-ps is determined by the interpretation of its axioms. In a cut-free indexed DiLL_0-ps, all the axioms can be interpreted by pairwise different atoms, which yields points of the relational interpretation that are more informative than others. We now define semantically such elements, called *injective.*

▶ **Definition 60** (Injective elements)**.** Let A be a MELL formula.
An element $a \in |A|$ is *injective* if every atom occuring in a occurs exactly twice.
If $X \subseteq |A|$, we set $X_{\text{inj}} = \{a \in \mathcal{X} \mid a \text{ is injective}\}$.

For example, $[\,] \in |!A|_{\text{inj}} = |?A|_{\text{inj}}$ for any formula A.

▶ **Definition 61** (Injective interpretation)**.** Let R be an indexed DiLL_0-proof structure. We define its *injective interpretation* as the set of injective points of its semantics:

$$[\![R]\!]_{\text{inj}} = [\![R]\!] \cap |\,⅋\Gamma\,|_{\text{inj}}$$

where Γ is the type of the conclusions of R.

▶ **Definition 62.** Let A be a MELL formula. We define $\sim_A$ as the equivalence relation on $|A|$ defined by: $a \sim_A a'$ iff there exists a bijection $\sigma : \mathcal{A}t \to \mathcal{A}t$ such that $a = \sigma_A(a')$.

▶ **Remark 63.** Let $A \in \mathcal{F}_{\mathsf{MELL}}$. If $a \in |A|_{\text{inj}}$ and $\sigma : \mathcal{A}t \to \mathcal{A}t$, then $\sigma_A(a) \in |A|_{\text{inj}}$.

Given a MELL formula A , the equivalence relation $\sim_A$ on $|A|_{\text{inj}}$ identifies any two injective points of A that are equal up to renaming of their atoms.

▶ **Proposition 64.** *Let* R *be an indexed cut-free and* η*-expanded* DiLL*-ps with conclusions of type* Γ*. The quotient of the identity* $[\![R]\!]_{\text{inj}} = \bigcup_{\overline{\rho}\in\mathcal{T}^{\simeq}_R}[\![\overline{\rho}]\!]_{\text{inj}}$ *through the equivalence* $\sim_{⅋\Gamma}$ *is a bijection* $[\![R]\!]_{\text{inj}}/\!\sim_{⅋\Gamma} \simeq \mathcal{T}^{\simeq}_R$.

We can thus say that the injective elements of $[\![R]\!]$ are the elements of the Taylor expansion of R, up to the renaming of the atoms.

This is folklore, as relational experiments and differential nets inside the Taylor expansion are variations around the idea that a box enshrines a subnet that can be copied an arbitrary number of time. We nonetheless decide to emphasize its precise formulation.